

Problem Set 1

Due: Sept. 3, 2008

1. A second-rank tensor is *symmetric* if it is unchanged when you interchange the indices ($s^{\nu\mu} = s^{\mu\nu}$) and it is *antisymmetric* if it changes sign ($a^{\nu\mu} = -a^{\mu\nu}$).
 - (a) How many independent elements are there in a symmetric tensor? (Since $s^{21} = s^{12}$, this counts as only *one* independent element.) In an antisymmetric tensor?
 - (b) If $s^{\mu\nu}$ is symmetric, and $a^{\mu\nu}$ is anti-symmetric, show that $s^{\mu\nu}a_{\mu\nu} = 0$.
 - (c) Show that any second-rank tensor $t^{\mu\nu}$ can be written as the sum of an anti-symmetric part ($a^{\mu\nu}$) and a symmetric part ($s^{\mu\nu}$): $t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$. Construct $a^{\mu\nu}$ and $s^{\mu\nu}$ explicitly, given $t^{\mu\nu}$.

2. An important tensor is the rank-4 tensor: $\epsilon^{\mu\nu\rho\sigma}$. This is defined to be completely antisymmetric (it changes sign under the interchange of any two indices, and is zero if any two are the same), with the definition that $\epsilon^{0123} = +1$. Under a Lorentz transformation, this will transform as

$$(\epsilon')^{\mu\nu\rho\sigma} = \epsilon^{\mu'\nu'\rho'\sigma'} \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} \Lambda^{\rho}_{\rho'} \Lambda^{\sigma}_{\sigma'}$$

Determine, in terms of the original tensor ϵ , what ϵ' is with the following Lorentz transformations:

- (a) A Lorentz boost in the 3-direction with speed v .
 - (b) A parity transformation, where $\Lambda^{\mu}_{\mu'} = P^{\mu}_{\mu'} = \text{diag}(1, -1, -1, -1)$.
 - (c) A time-reversal transformation, where $\Lambda^{\mu}_{\mu'} = T^{\mu}_{\mu'} = \text{diag}(-1, 1, 1, 1)$.
3. Particle A , at rest, decays as follows: $A \rightarrow \sum_{i=1}^N B_i$, with $N \geq 3$.
 - (a) Determine the maximum and minimum energies that B_1 can have in such a decay, in terms of the various masses.
 - (b) What is the maximum energy of the electron in the basic beta decay reaction $n \rightarrow p + e^- + \bar{\nu}_e$, assuming the neutrino is massless?

4. Show that if \mathcal{L} is a Lorentz scalar, that is

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}$$

under a Lorentz boost, that the action $S = \int d^4x \mathcal{L}$ is Lorentz invariant. The question here is how do volumes change under a Lorentz boost?

5. A useful theoretical set of kinematic invariants are the *Mandelstam variables*, defined for a $2 \rightarrow 2$ body scattering process: $A + B \rightarrow C + D$,

$$\begin{aligned} s &= (p_A + p_B)^2 = (p_C + p_D)^2 \\ t &= (p_A - p_C)^2 = (p_B - p_D)^2 \\ u &= (p_A - p_D)^2 = (p_B - p_C)^2 \end{aligned}$$

- (a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.
- (b) Determine in the CM frame, s in terms of the CM energy $E_{CM} = E_A + E_B$
- (c) In the CM frame in the special case of all four masses being identical, determine t and u in terms of the three-momentum and the scattering angle θ_{CM} , defined by the angle between A and say, C .

- (d) Now for the case of elastic scattering ($A + B \rightarrow A + B$) in the lab frame ($\mathbf{p}_B = 0$), determine s , t , and u . Note that your expressions for t and u will be complicated, so you can leave them in terms of the incoming and outgoing A energies as well as the angle between the incoming and outgoing A 's. Just include all equations (coming from conservation of momentum/energy) that would allow you to implicitly write s, t, u in terms of just two variables.
6. Construct higher spin states for $SU(2)$, as has been done in Quantum Mechanics for the spin-1/2 state. This will be useful not just for angular momentum, but for when we discuss isospin. For spin 1 we have three states ($m_s = +1, 0, -1$), which we may represent by column vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. The only problem is to construct the 3×3 matrices S_1, S_2, S_3 . The latter is easy:

- (a) Construct S_3 for spin 1.
- (b) To obtain $S_{1,2}$, it is easiest to start with the “raising” and “lowering” operators, $S_{\pm} = S_1 \pm iS_2$, which have the property

$$S_{\pm}|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}|s, (m_s \pm 1)\rangle$$

Construct the matrices S_+ and S_- for spin 1.

- (c) Now determine S_1 and S_2 .
- (d) Carry out the same construction for spin 3/2.