Problem Set 1 Solutions

Due: Sept. 3, 2008

1. A second-rank tensor is *symmetric* if it is unchanged when you interchange the indices ($s^{\nu\mu} = s^{\mu\nu}$)and it is *antisymmetric* if it changes sign ($a^{\nu\mu} = -a^{\mu\nu}$).

   (a) How many independent elements are there in a symmetric tensor? (Since $s^{21} = s^{12}$, this counts as only *one* independent element.) In an antisymmetric tensor?

   (b) If $s^{\mu\nu}$ is symmetric, and $a^{\mu\nu}$ is anti-symmetric, show that $s^{\mu\nu}a_{\mu\nu} = 0$.

   (c) Show that any second-rank tensor $t^{\mu\nu}$ can be written as the sum of an anti-symmetric part ($a^{\mu\nu}$) and a symmetric part ($s^{\mu\nu}$): $t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$. Construct $a^{\mu\nu}$ and $s^{\mu\nu}$ explicitly, given $t^{\mu\nu}$.

Solution:

(a) In a symmetric tensor, we have four diagonal components that are independent, and six off-diagonal components (half of the twelve total), so we have **10** independent components in four dimensions.

In an arbitrary number of n dimensions, we have n diagonal elements, and $n^2 - n$ off-diagonal elements (half of which are independent), so we have in general

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

independent components.

For an antisymmetric tensor, we have the same number of off-diagonal components (since $a^{\mu\nu} = -a^{\nu\mu}$, these are not independent), and all the diagonal components must be zero, so we have **6** (or $(n^2 - n)/2$ in general) independent components.

(b) We can write this out, putting the explicit summations in:

$$\begin{aligned}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} s^{\mu\nu}a_{\mu\nu} &= \sum_{\mu=0}^{3} s^{\mu\mu}a_{\mu\mu} + \sum_{\mu>\nu}^{3} s^{\mu\nu}a_{\mu\nu} + \sum_{\mu<\nu}^{3} s^{\mu\nu}a_{\mu\nu} \\ &= \sum_{\mu>\nu}^{3} s^{\mu\nu}a_{\mu\nu} + \sum_{\nu<\mu}^{3} s^{\nu\mu}a_{\nu\mu}\end{aligned}$$

We got rid of the first term because the diagonal elements of a are zero, and we interchanged μ, ν in the last term, which is allowed because these are dummy indices. Using the fact that $s^{\mu\nu} = s^{\nu\mu}$ and $a^{\mu\nu} = -a^{\nu\mu}$, we can see that this final term equals the opposite of the first term, thus

$$s^{\mu\nu}a_{\mu\nu} = 0$$

(c) Since a second-rank tensor has n^2 (or 16 in four dimensions) free components, then we can see that with the $n(n+1)/2$ components in s, and the $n(n-1)/2$ components in a, then the right-hand side of

$$t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$$

has the same number of independent parameters as the left-hand side. We can also write, using the symmetry properties of s, a:

$$t^{\nu\mu} = s^{\mu\nu} - a^{\mu\nu}$$

so that

$$s^{\mu\nu} = \frac{1}{2}\left[t^{\mu\nu} + t^{\nu\mu}\right]$$

$$a^{\mu\nu} = \frac{1}{2}\left[t^{\mu\nu} - t^{\nu\mu}\right]$$

2. An important tensor is the rank-4 tensor: $\epsilon^{\mu\nu\rho\sigma}$. This is defined to be completely antisymmetric (it changes sign under the interchange of any two indices, and is zero is any two are the same), with the definition that $\epsilon^{0123} = +1$. Under a Lorentz transformation, this will transform as

$$(\epsilon')^{\mu\nu\rho\sigma} = \epsilon^{\mu'\nu'\rho'\sigma'} \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \Lambda^{\rho}{}_{\rho'} \Lambda^{\sigma}{}_{\sigma'}$$

Determine, in terms of the orginal tensor ϵ, what ϵ' is with the following Lorentz transformations:

(a) A Lorentz boost in the 3-direction with speed v.

(b) A parity transformation, where $\Lambda^{\mu}{}_{\mu'} = P^{\mu}{}_{\mu'} = \text{diag}(1, -1, -1, -1)$.

(c) A time-reversal transformation, where $\Lambda^{\mu}{}_{\mu'} = T^{\mu}{}_{\mu'} = \text{diag}(-1, 1, 1, 1)$.

Solution:

(a) A Lorentz boost in the 3-direction has the form

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix}$$

This is symmetric under $\mu \leftrightarrow \nu$ interchange, while the epsilon tensor is antisymmetric under the interchange of any two indices. This means that ϵ' is also antisymmetric under the interchange of any two indices, as we can see for example, with $\mu \leftrightarrow \nu$:

$$\begin{aligned} (\epsilon')^{\nu\mu\rho\sigma} &= \epsilon^{\mu'\nu'\rho'\sigma'} \Lambda^{\nu}{}_{\mu'} \Lambda^{\mu}{}_{\nu'} \Lambda^{\rho}{}_{\rho'} \Lambda^{\sigma}{}_{\sigma'} \\ &= \epsilon^{\nu'\mu'\rho'\sigma'} \Lambda^{\nu}{}_{\nu'} \Lambda^{\mu}{}_{\mu'} \Lambda^{\rho}{}_{\rho'} \Lambda^{\sigma}{}_{\sigma'} \\ &= -\epsilon^{\mu'\nu'\rho'\sigma'} \Lambda^{\nu}{}_{\nu'} \Lambda^{\mu}{}_{\mu'} \Lambda^{\rho}{}_{\rho'} \Lambda^{\sigma}{}_{\sigma'} = -(\epsilon')^{\mu\nu\rho\sigma} \end{aligned}$$

Thus, if we determine just one component of ϵ', we will know all of them.
We can determine ϵ' for $\mu\nu\rho\sigma = 0123$ easily:

$$(\epsilon')^{0123} = \epsilon^{\mu'\nu'\rho'\sigma'} \Lambda^{0}{}_{\mu'} \Lambda^{1}{}_{\nu'} \Lambda^{2}{}_{\rho'} \Lambda^{3}{}_{\sigma'}$$

For $\Lambda^{1}{}_{\nu'}, \Lambda^{2}{}_{\rho'}$, there is only one non-zero entry, and two for the others. Writing these out:

$$\begin{aligned} (\epsilon')^{0123} &= \epsilon^{012\sigma'} \Lambda^{0}{}_{0} \Lambda^{3}{}_{\sigma'} + \epsilon^{312\sigma'} \Lambda^{0}{}_{3} \Lambda^{3}{}_{\sigma'} \\ &= \epsilon^{0123} \Lambda^{0}{}_{0} \Lambda^{3}{}_{3} + \epsilon^{3120} \Lambda^{0}{}_{3} \Lambda^{3}{}_{0} \\ &= \epsilon^{0123} \left[\Lambda^{0}{}_{0} \Lambda^{3}{}_{3} - \Lambda^{0}{}_{3} \Lambda^{3}{}_{0} \right] \\ &= \epsilon^{0123} \left[\gamma^2 - (-\gamma v)^2 \right] = \epsilon^{0123} \gamma^2 \left[1 - v^2 \right] = \epsilon^{0123} \end{aligned}$$

Thus, under a boost:

$$(\epsilon')^{\nu\mu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}$$

(Note: this is also true for any boost *and* any rotation.)

(b) For a parity transformation, we only have the diagonal elements:

$$(\epsilon')^{0123} = \epsilon^{\mu'\nu'\rho'\sigma'} P^{0}{}_{\mu'} P^{1}{}_{\nu'} P^{2}{}_{\rho'} P^{3}{}_{\sigma'} = \epsilon^{0123}(+1)(-1)(-1)(-1) = -\epsilon^{0123}$$

So this tensor is *odd* under parity.

(c) For time-reversal, we have $T^{\mu}{}_{\nu} = -P^{\mu}{}_{\nu}$, so we can take the result of the previous part and get

$$(\epsilon')^{\nu\mu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$$

3. Particle A, at rest, decays as follows: $A \rightarrow \sum_{i=1}^{N} B_i$, with $N \geq 3$.

   (a) Determine the maximum and minimum energies that B_1 can have in such a decay, in terms of the various masses.

   (b) What is the maximum energy of the electron in the basic beta decay reaction $n \rightarrow p + e^- + \bar{\nu}_e$, assuming the neutrino is massless?

Solution:

(a) The minimum energy of B_1 is rather trivial, as it could have zero momentum, and thus

$$E_{\text{min}} = m_1$$

As for the maximum energy, this happens when B_n for $n \geq 2$ are all moving in the same direction, exactly opposite the direction of B_1. If any of the particles have motion in the directions perpendicular to the direction of B_1, we have to put energy into that motion.
We can define the four-momentum of the $i \geq 2$ particles as

$$P = \sum_{i \geq 2} p_i \equiv (E, \mathbf{P})$$

and this satisfies

$$P^2 = M^2 = \left(\sum_{i \geq 2} m_i \right)^2$$

and $\mathbf{P} = -\mathbf{p}_1$. Also, from energy conservation:

$$m_A = E + E_1 \ .$$

Now this means that

$$\begin{aligned}
p_A^2 &= (p_1 + P)^2 \\
m_A^2 &= m_1^2 + M^2 + 2 p_1 \cdot P \\
m_A^2 &= m_1^2 + M^2 + 2(E_1 E + \mathbf{p}_1^2) \\
m_A^2 &= m_1^2 + M^2 + 2(E_1(m_A - E_1) + E_1^2 - m_1^2) \\
m_A^2 &= -m_1^2 + M^2 + 2 m_A E_1
\end{aligned}$$

or

$$E_{1,\text{max}} = \frac{m_A^2 - M^2 + m_1^2}{2 m_A}$$

(b) If the neutrino is massless, then we have $M = m_p$, so we would expect

$$E_{e,\text{max}} = \frac{m_n^2 - m_p^2 + m_e^2}{2 m_n} = 1.2926 \text{ MeV}$$

4. Show that if $\mathcal{L}$ is a Lorentz scalar, that is

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}$$

under a Lorentz boost, that the action $S = \int d^4x \mathcal{L}$ is Lorentz invariant. The question here is how do volumes change under a Lorentz boost?

Solution:

What is important here is to realize that we can write a Lorentz transformation as

$$\Lambda^\mu{}_\nu = \frac{d(x')^\mu}{dx^\nu}$$

so it is a change of basis for spacetime variables. Under a Lorentz transformation, we have

$$S = \int d^4x\mathcal{L} \to \int d^4x'\mathcal{L}$$

The matrix describing this change of basis is precisely Λ and we know that

$$\int d^4x' = \int d^4x|J|$$

where J is the Jacobian, given by the determinant of Λ. This of course was shown to be ± 1, and thus

$$\int d^4x' = \int d^4x$$

which is precisely what we wanted to prove.

5. A useful theoretical set of kinematic invariants are the *Mandelstam variables*, defined for a $2 \to 2$ body scattering process: $A + B \to C + D$,

$$\begin{aligned} s &= (p_A + p_B)^2 = (p_C + p_D)^2 \\ t &= (p_A - p_C)^2 = (p_B - p_D)^2 \\ u &= (p_A - p_D)^2 = (p_B - p_C)^2 \end{aligned}$$

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

(b) Determine in the CM frame, s in terms of the CM energy $E_{CM} = E_A + E_B$

(c) In the CM frame in the special case of all four masses being identical, determine t and u in terms of the three-momentum and the scattering angle θ_{CM}, defined by the angle between A and say, C.

(d) Now for the case of elastic scattering ($A + B \to A + B$) in the lab frame ($\mathbf{p}_B = 0$), determine $s, t,$ and u. Note that your expressions for t and u will be complicated, so you can leave them in terms of the incoming and outgoing A energies as well as the angle between the incoming and outgoing A's. Just include all equations (coming from conservation of momentum/energy) that would allow you to implicitly write s, t, u in terms of just two variables.

Solution:

(a) This we can do by simple substitution:

$$\begin{aligned} s + t + u &= (p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2 \\ &= m_A^2 + m_B^2 + 2p_A \cdot p_B + m_A^2 + m_C^2 - 2p_A \cdot p_C + m_A^2 + m_D^2 - 2p_A \cdot p_D \\ &= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2p_A \cdot (p_B - p_C - p_D) \\ &= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2p_A \cdot (-p_A) \\ &= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2m_A^2 = m_A^2 + m_B^2 + m_C^2 + m_D^2 \end{aligned}$$

(b) In the CM frame, we have the three momentum given by

$$\mathbf{p}_A = -\mathbf{p}_B$$

so the four-momentum

$$p_A + p_B = (E_A + E_B, 0)$$

and

$$s = (E_A + E_B)^2 = E_{CM}^2$$

(c) If all the masses are equal, then all the momenta have the same magnitude $|\mathbf{p}|$, and they all have the same energy (so the zero components of $p_A - p_C = p_A - p_D$ are zero). t is given by

$$\begin{aligned} t &= (p_A - p_C)^2 \\ &= -(\mathbf{p}_A - \mathbf{p}_C)^2 \\ &= -|\mathbf{p}|^2 - |\mathbf{p}|^2 - 2|\mathbf{p}|^2 \cos\theta \\ &= -2|\mathbf{p}|^2(1 + \cos\theta) \end{aligned}$$

We can easily calculate u because if the angle between A and C is θ, then that between A and D is $\theta - \pi$, so

$$u = -2|\mathbf{p}|^2(1 - \cos\theta)$$

(d) We have the momenta, say

$$\text{initial: } p_A = (E, 0, 0, p) \;, \quad p_B = (m_B, \mathbf{0})$$

$$\text{final: } p_A = (E', p'\sin\theta, 0, p'\cos\theta) \;, \quad p_B = (E_B, -p_B\sin\phi, 0, p_B\cos\phi)$$

We have the following equations from energy-momentum conservation:

$$E + m_B = E' + E_B$$

$$p'\sin\theta = p_B \sin\phi$$

$$p = p'\cos\theta + p_B\sin\phi$$

So first:

$$s = (p_A + p_B)^2 = (E + m_B)^2 - p^2 = E^2 + m_B^2 + 2Em_B - p^2 = m_A^2 + m_B^2 + 2Em_B$$

For t we have the difference in the initial and final A momenta squared (notice that there is a bit of an ambiguity in the definition of t and u, and generally we define t to be the difference of the four momenta of the "most similar" particles)

$$t = (p_A^i - p_A^f)^2 = (E' - E)^2 - (p')^2(\sin^2\theta + \cos^2\theta) = (E')^2 + E^2 - 2E'E - (p')^2$$

or

$$t = m_A^2 + E^2 - 2E'E$$

but recall that ultimately E' is a function of the angles ϕ, θ, given by the three equations above (combined with the 3 energy-momentum relations). We will leave the equations in this implicit form.

For u, we could calculate it directly or use

$$s + t + u = 2(m_A^2 + m_B^2)$$

so

$$u = (m_B - E)^2 + 2E'E$$

6. Construct higher spin states for $SU(2)$, as has been done in Quantum Mechanics for the spin-1/2 state. This will be useful not just for angular momentum, but for when we discuss isospin. For spin 1 we have three states ($m_s = +1, 0, -1$), which we may represent by column vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. The only problem is to construct the 3×3 matrices S_1, S_2, S_3. The latter is easy:

(a) Construct S_3 for spin 1.

(b) To obtain $S_{1,2}$, it is easiest to start with the "raising" and "lowering" operators, $S_\pm = S_1 \pm iS_2$, which have the property

$$S_\pm|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}|s, (m_s \pm 1)\rangle$$

Construct the matrices S_+ and S_- for spin 1.

(c) Now determine S_1 and S_2.

(d) Carry out the same construction for spin 3/2.

Solution:

(a) Since we work in a basis that has S_3 diagonal, then it must be composed of its eigenvalues:

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) For $S_\pm$, we start from the $m_s = +1$ state, and lower it:

$$S_-|1, +1\rangle = \sqrt{2}|1, 0\rangle$$

and again

$$S_-|1, 0\rangle = \sqrt{2}|1, -1\rangle$$

and finally if we act S_- on $|1, -1\rangle$, we get 0. Using the basis of vectors above, we get the matrix

$$S_- = \begin{pmatrix} & & \\ \sqrt{2} & & \\ & \sqrt{2} & 0 \end{pmatrix}$$

The empty spaces can be anything, and we'll set them to zero. S_+ can be found just by taking the dagger of this, so we have

$$S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

(c) From these, we get

$$S_1 = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$S_2 = \frac{1}{2i}(S_+ - S_-) = \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(d) Doing this for the spin-3/2 case is very straightforward. First

$$S_3 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

We have the eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and we can calculate the $S_\pm$ matrices. We have

$$S_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and then

$$S_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$S_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and

$$S_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$