Problem Set 2

Due: Sept. 10, 2008

1. Show that for $SU(2)$, the doublet of nucleon antiparticles $\overline{N}$ transforms the same was as N under isospin transformations, where
$$\overline{N} = \begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix}, \qquad N = \begin{pmatrix} p \\ n \end{pmatrix}$$
In other words, if $N \to UN$ with $U \in SU(2)$, then $\overline{N} \to U\overline{N}$, with U the same matrix. [Technically n, p are spinors, but you can treat $\overline{n}$ as n^*, and the same with $\overline{p}$. However, they still are not one-component objects, so $1/n$ is meaningless.]

2. The following refers to a general Lie algebra
$$[T^a, T^b] = if^{abc}T^c$$
where T^a are the generators of the group. For $SU(2)$, $T^a = \sigma^a/2$ and $f^{abc} = \epsilon^{abc}$.
   (a) Show that the f^{abc} are completely antisymmetric under the interchange of any two indices.
   (b) Define a set of matrices t^a, such that the components are
$$(t^a)_{bc} = -if^{abc}$$
   Using the commutator above and the Jacobi Identity
$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \ ,$$
   show that these matrices also form a representation of the group, or
$$[t^a, t^b] = if^{abc}t^c \ .$$
   This is known as the adjoint representation of the group.
   (c) Given that the dimension of the fundamental representation of the group $SU(N)$ is N, what is the dimension of the adjoint representation of the group? [Work this out for the explicit examples of $SU(2)$ and $SU(3)$ and look for a pattern, you can then figure it out from there.]

3. The Lorentz group, or more specifically, the subgroup of the Lorentz group which is *continuously connected to the identity* (ie, rotations and boosts), can be divided in a very specific way. It helps to write down the defining relationship for the generators of the group. If $\Lambda = \exp[-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}]$, then one can show that
$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho})$$
Define the generators of rotations and boosts, respectively, as
$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i} \ ,$$
with $i, j, k = 1, 2, 3$. Under an infinitessimal transformation, ω^{ij} corresponds to angles about a given axis and ω^{0i} corresponds to boosts in the ith direction. Write down the three different sets of commutation relations for these vector operators (for example, $[L^i, L^j] = i\epsilon^{ijk}L^k$). Show that the combinations
$$\mathbf{A} = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \text{ and } \mathbf{B} = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$
commute with each other and separately satisfy the commutation relations of angular momentum.

This implies that any representation of the Lorentz group $SO(3,1)$ can be written down in terms of how it transforms under $SU(2)_A \times SU(2)_B$, using the notation (j_A, j_B). Particles are defined by their transformation properties under the Lorentz group, and this is just the same as how they transform under a direct product of $SU(2)$ groups.

4. Griffiths 2.7. This has a lot of parts, but should be fairly quick to get through. You will need to consider for many cases the quark level diagrams for the hadrons as shown in the text.

5. *Additional problem (Not something for you to turn in, but you'll be expected to understand this)*: What is the OZI rule, and how does it allow for the long lifetime of the J/ψ?