Problem Set 2 Solutions

Due: Sept. 10, 2008

1. Show that for $SU(2)$, the doublet of nucleon antiparticles $\overline{N}$ transforms the same was as N under isospin transformations, where
$$\overline{N} = \begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix}, \qquad N = \begin{pmatrix} p \\ n \end{pmatrix}$$
In other words, if $N \to UN$ with $U \in SU(2)$, then $\overline{N} \to U\overline{N}$, with U the same matrix. [Technically n, p are spinors, but you can treat $\overline{n}$ as n^*, and the same with $\overline{p}$. However, they still are not one-component objects, so $1/n$ is meaningless.]

Solution:

Since we can treat $\bar{n}$ as n^*, We can write $\bar{N}$ in terms of N:

$$\begin{aligned}
\overline{N} &= \begin{pmatrix} -\overline{n} \\ \overline{p} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{p} \\ \overline{n} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} N^* \\
&= -i\tau^2 N^*
\end{aligned}$$

where τ^2 is the second Pauli matrix, and we denote it with τ so as to not confuse it with the Pauli spin matrix σ^2.

So under an $SU(2)$ transformation, we have

$$\begin{aligned}
\overline{N} &= -i\tau^2 N^* \\
&\to -i\tau^2 (UN)^* \\
&= -i\tau^2 U^* N^*
\end{aligned}$$

(Note we are not taking the Hermitian conjugate, just the complex conjugate!) U can be written in terms of the generators of $SU(2)$:
$$U = \exp[-i\theta^i \tau^i/2]$$
so
$$U^* = \exp[+i\theta^i (\tau^i)^*/2]$$
and we can use the relation that
$$\tau^2 \tau^i \tau^2 = -(\tau^i)^*$$
or
$$\tau^i \tau^2 = -\tau^2 (\tau^i)^*$$
(this can be worked out explicitly for each value of i). By expanding the exponential U^* as a Taylor series, we can move the Pauli matrix to the right of U^* to get

$$\tau^2 U^* = \tau^2 \exp[+i\theta^i (\tau^i)^*/2] = \exp[-i\theta^i \tau^i/2]\tau^2 = U\tau^2$$

and so

$$\overline{N} \to U(-i\tau^2)N^* = U\overline{N}$$

which is what we wanted to show.

2. The following refers to a general Lie algebra

$$[T^a, T^b] = i f^{abc} T^c$$

where T^a are the generators of the group. For $SU(2)$, $T^a = \sigma^a/2$ and $f^{abc} = \epsilon^{abc}$.

(a) Show that the f^{abc} are completely antisymmetric under the interchange of any two indices.

(b) Define a set of matrices t^a, such that the components are

$$(t^a)_{bc} = -i f^{abc}$$

Using the commutator above and the Jacobi Identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \ ,$$

show that these matrices also form a representation of the group, or

$$[t^a, t^b] = i f^{abc} t^c \ .$$

This is known as the adjoint representation of the group.

(c) Given that the dimension of the fundamental representation of the group $SU(N)$ is N, what is the dimension of the adjoint representation of the group? [Work this out for the explicit examples of $SU(2)$ and $SU(3)$ and look for a pattern, you can then figure it out from there.]

Solution:

(a) It is trivial to see that

$$f^{bac} = -f^{abc}$$

due to the antisymmetry of the commutator. But instead we would like to show this is true for all indices. We do this by multiplying both sides by an additional generator T^d, and taking the trace of both sides. This gives us

$$\mathrm{Tr}\left\{[T^a, T^b]T^d\right\} = i f^{abc}\,\mathrm{Tr}[T^c T^d]$$

Using the normalization of the generators to be such that

$$\mathrm{Tr}[T^c T^d] = \frac{1}{2}\delta^{cd}$$

(Note this is only true for the fundamental, but for other representations the 1/2 just changes to another constant, and we really only care about the indices.) Thus we have

$$\mathrm{Tr}\left\{T^a T^b T^d\right\} - \mathrm{Tr}\left\{T^b T^a T^d\right\} = \frac{i}{2} f^{abd}$$

Let's take this and interchange $a \leftrightarrow d$. We get

$$\frac{i}{2} f^{dba} = \mathrm{Tr}\left\{T^d T^b T^a\right\} - \mathrm{Tr}\left\{T^b T^d T^a\right\}$$

But the trace is cyclic, so I can move T^d in the first term to the end and T^a in the second term to the beginning to get

$$\frac{i}{2} f^{dba} = \mathrm{Tr}\left\{T^b T^a T^d\right\} - \mathrm{Tr}\left\{T^a T^b T^d\right\} = -\frac{i}{2} f^{abd}$$

Thus $f^{dba} = -f^{abd}$. We can use the same expression to show this is true for the last interchange of indices and see that $f^{abd} = -f^{adb}$, and thus this is completely antisymmetric under the interchange of all indices. Note this is dependent on us being able to normalize the generators so that

$$\mathrm{Tr}[T^c T^d] = C(r)\delta^{cd}$$

where $C(r)$ is a constant that only depends on the representation. This can be shown to always be possible.

(b) Well the Jacobi identity gives us

$$\begin{aligned}
[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]] &= i\left\{f^{bcd}[T^a,T^d]+f^{cad}[T^b,T^d]+f^{abd}[T^c,T^d]\right\} \\
&= (i)(i)\left\{f^{bcd}f^{ade}+f^{cad}f^{bde}+f^{abd}f^{cde}\right\}T^e
\end{aligned}$$

This is true for all T^e, so

$$f^{bcd}f^{ade}+f^{cad}f^{bde}+f^{abd}f^{cde}=0$$

(which is also referred to as the Jacobi identity sometimes). Let's write the first term in terms of the adjoint generators:

$$i(t^b)_{cd}i(t^a)_{de}+f^{cad}f^{bde}+f^{abd}f^{cde}=0$$

This is just a matrix product of t^a, t^b. By rearranging the indices in the first structure constant of the second term (picking up a minus sign), that can also look like the same matrix product in the reverse order:

$$i(t^b)_{cd}i(t^a)_{de}-i(t^a)_{cd}i(t^b)_{de}+f^{abd}f^{cde}=0$$

This can be written now as (note the indices)

$$[t^a,t^b]_{ce}=-f^{abd}f^{cde}$$

or rearranging the cde indices and turning that into a generator, we have

$$[t^a,t^b]_{ce}=+f^{abd}f^{dce}=if^{abd}(t^d)_{ce}$$

which shows that these are generators of the group. dropping indices:

$$[t^a,t^b]=if^{abd}t^d$$

(c) The dimension of the fundamental representation is N for $SU(N)$, but what about the adjoint? Well, for $SU(2)$, we have $N=2$ and the dimension of the adjoint representation is 3. This comes from the fact that there are three generators of $SU(2)$, and the adjoint generators are defined to be matrices with elements

$$(t^a)_{bc}$$

For $SU(2)$, b, c all run from 1 to 3, so these are 3×3 matrices.

In general though, we need to know how many independent types of "rotations" we can make for arbitrary N (note these are not rotations in the same sense as in $SU(2)$ spin, but just group transformations). We can find this out using the fundamental represention. This representation transforms with matrices that are $N\times N$ unitary matrices with determinant one. In a general complex matrix, there are $2N^2$ independent elements. But for it to be unitary, we have

$$U^\dagger U=1$$

and there are N diagonal equations coming from this, of the form (sums are explicit now and repeated indices are not necessarily summed over)

$$\sum_k U^*_{ki}U_{ki}=\sum_k |U_{ki}|^2 1$$

Hre the RHS is the number one not the unit matrix, and note the order of the indices. Then there are the off-diagonal equations

$$\sum_k U^*_{ki}U_{kj}=0$$

and here there are $2(N^2-N)$ of them (N^2-N for the real and N^2-N for the imaginary components; this wasn't true for the diagonal because the LHS was a real number to begin with), only half of which are independent equations (because the other half are just the complex conjugate of the above equation), so we have

$$2N^2 - N - (N^2 - N) = N^2$$

independent parameters. But that is for $U(N)$, any $N \times N$ unitary matrix. The determinant must also be equal to one, which is another constraint, so we have

$$N^2 - 1$$

independent parameters, and thus $N^2 - 1$ generators. Finally, this must be the dimension of the adjoint representation.

3. The Lorentz group, or more specifically, the subgroup of the Lorentz group which is *continuously connected to the identity* (ie, rotations and boosts), can be divided in a very specific way. It helps to write down the defining relationship for the generators of the group. If $\Lambda = \exp[-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}]$, then one can show that

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho})$$

Define the generators of rotations and boosts, respectively, as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i} \ ,$$

with $i, j, k = 1, 2, 3$. Under an infinitessimal transformation, ω^{ij} corresponds to angles about a given axis and ω^{0i} corresponds to boosts in the ith direction. Write down the three different sets of commutation relations for these vector operators (for example, $[L^i, L^j] = i\epsilon^{ijk}L^k$). Show that the combinations

$$\mathbf{A} = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \text{ and } \mathbf{B} = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with each other and separately satisfy the commutation relations of angular momentum.

This implies that any representation of the Lorentz group $SO(3,1)$ can be written down in terms of how it transforms under $SU(2)_A \times SU(2)_B$, using the notation (j_A, j_B). Particles are defined by their transformation properties under the Lorentz group, and this is just the same as how they transform under a direct product of $SU(2)$ groups.

Solution:

We first make the comment that since all sums here are over *spatial* Lorentz indices, we needn't worry about whether or not they are raised or lowered; there are no sign ambiguities that arise. Also, there is no difference between $\mathcal{J}$ and J...

For the L^i, we have (there will be signs arising because all the metrics below are spatial, though)

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}[J^{kl}, J^{mn}] \\
&= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}i(g^{lm}J^{kn} - g^{km}J^{ln} - g^{ln}J^{km} + g^{kn}J^{lm}) \\
&= -\frac{1}{4}i(J^{kn}\epsilon^{ikl}\epsilon^{jln} - J^{ln}\epsilon^{ikl}\epsilon^{jkn} - J^{km}\epsilon^{ikl}\epsilon^{jml} + J^{lm}\epsilon^{ikl}\epsilon^{jmk})
\end{aligned}$$

Let's interchange $l \leftrightarrow k$ in term 1 and term 4, and we get

$$\begin{aligned}
[L^i, L^j] &= -\frac{1}{4}i(J^{ln}\epsilon^{ilk}\epsilon^{jkn} - J^{ln}\epsilon^{ikl}\epsilon^{jkn} - J^{km}\epsilon^{ikl}\epsilon^{jml} + J^{km}\epsilon^{ilk}\epsilon^{jml}) \\
&= +2\frac{1}{4}i(J^{ln}\epsilon^{ikl}\epsilon^{jkn} + J^{km}\epsilon^{ikl}\epsilon^{jml}) \\
&= \frac{1}{2}i(J^{km}\epsilon^{ilk}\epsilon^{jlm} + J^{km}\epsilon^{ikl}\epsilon^{jml}) \\
&= iJ^{km}\epsilon^{ilk}\epsilon^{jlm} \\
&= iJ^{km}(\delta^{mk}\delta^{ij} - \delta^{mi}\delta^{kj}) \\
&= -iJ^{ji}
\end{aligned}$$

The last equality follows because J is antisymmetric. We can now use the inverted form of the definition of L in terms of J:

$$J^{ji} = \epsilon^{jik} L^k = -\epsilon^{ijk} L^k$$

to get

$$[L^i, L^j] = i\epsilon^{ijk} L^k$$

These used the relationship

$$\epsilon^{mni}\epsilon^{ijk} = (\delta^{mj}\delta^{nk} - \delta^{nj}\delta^{mk})$$

multiple times.

Now for the boosts (remembering that the metric is diagonal and J is antisymmetric):

$$\begin{aligned}
[K^i, K^j] &= [J^{0i}, J^{0j}] \\
&= i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0})) \\
&= -iJ^{ij}
\end{aligned}$$

or

$$[K^i, K^j] = -i\epsilon^{ijk} L^k$$

So notice that effectively, two rotations will correspond to another rotation (those transformations are closed on themselves), but two boosts in different directions would correspond to a rotation, not a boost.

As for the final commutator, we have

$$\begin{aligned}
[L^i, K^j] &= \frac{1}{2}\epsilon^{ikm}[J^{km}, J^{0j}] \\
&= i\frac{1}{2}\epsilon^{ikm}(g^{m0}J^{kj} - g^{k0}J^{mj} - g^{mj}J^{k0} + g^{kj}J^{m0}) \\
&= i\frac{1}{2}(-g^{mj}J^{k0}\epsilon^{ikm} + g^{kj}J^{m0}\epsilon^{ikm}) \\
&= i\frac{1}{2}(+J^{k0}\epsilon^{ikj} - J^{m0}\epsilon^{ijm}) \\
&= iJ^{k0}\epsilon^{ikj}
\end{aligned}$$

or

$$[L^i, K^j] = i\epsilon^{ijk} K^k$$

so a rotation and then a boost in a different direction just corresponds to a boost in the third direction. The main lesson is that these different generators mix among themselves so we cannot define particles by their transformations under rotations or boosts alone.

Using these commutation relations we can work out those for A^i and B^i. Let's for now call them $A^i_+ = A_i$ and $A^i_- = B^i$, so we can do the $[A^i, A^j]$ and $[B^i, B^j]$ commutators simultaneously:

$$\begin{aligned}
[A^i_\pm, A^j_\pm] &= \frac{1}{4}\left\{[L^i, L^j] \pm i[L^i, K^j] \pm i[K^i, L^j] - [K^i, K^j]\right\} \\
&= \frac{1}{4}i\left\{\epsilon^{ijk}L^k \pm i\epsilon^{ijk}K^k \pm i\epsilon^{ijk}K^k + \epsilon^{ijk}L^k\right\} \\
&= \frac{1}{2}i\epsilon^{ijk}\left\{L^k \pm iK^k\right\}
\end{aligned}$$

Or

$$[A^i, A^j] = i\epsilon^{ijk} A^k$$

$$[B^i, B^j] = i\epsilon^{ijk} B^k$$

As for the commutator between them, we get

$$\begin{aligned}
[A^i, B^j] &= \frac{1}{4}\left\{[L^i, L^j] - i[L^i, K^j] + i[K^i, L^j] + [K^i, K^j]\right\} \\
&= \frac{1}{4} i\epsilon^{ijk}\left\{L^k - iK^k + iK^k + (-L^k)\right\} \\
&= 0
\end{aligned}$$

So they commute. This means that A and B form independent subgroups of $SU(2)$, as they each satisfy the algebra of $SU(2)$. Thus all states that transform under the Lorentz group will transform one way under the "A" $SU(2)$ and some other way under the "B" $SU(2)$.

4. Griffiths 2.7. This has a lot of parts, but should be fairly quick to get through. You will need to consider for many cases the quark level diagrams for the hadrons as shown in the text.

Solution:

(a) $p + \overline{p} \to \pi^+ + \pi^0$: No, violates charge conservation.

(b) $\eta \to \gamma + \gamma$: Yes, by way of the electromagnetic interaction; the η is made up of a linear combination of $\bar{u}u, \bar{d}d$, and $\bar{s}s$, and so the underlying interaction is $q + \bar{q} \to 2\gamma$.

(c) $\Sigma^0 \to \Lambda + \pi^0$: No, this violates energy conservation. $m_{\Sigma^0} = 1193$, while $m_\Lambda + m_{\pi^0} = 1116 + 135 = 1251$ MeV.

(d) $\Sigma^- \to n + \pi^-$: Yes, by way of the weak interactions. The underlying quark process is $dds \to udd + \bar{u}d$, so we have to turn a strange quark into a down quark.

(e) $e^+ + e^- \to \mu^+ + \mu^-$: Yes, by either electromagnetic or weak interactions (there can be an intermediate Z boson or a photon), but of course unless we are near the Z^0 mass, the electromagnetic interaction dominates.

(f) $\mu^- \to e^- + \bar{\nu}_e$: No, violates muon Lepton number.

(g) $\Delta^+ \to p + \pi^0$: Yes, by way of the electromagnetic or neutral weak interaction. The underlying process is a uud combination emitting a photon or Z, which then decays to a $\bar{u}u$ or $\bar{d}d$ pair.

(h) $\bar{\nu}_e + p \to n + e^+$: Yes, via the weak interactions (this is a form of inverse beta decay).

(i) $e + p \to \nu_e + \pi^0$: No, violates baryon number.

(j) $p + p \to \Sigma^+ + n + K^0 + \pi^+ + \pi^0$: Yes, via the strong interactions. The underlying process is $uud + uud \to uus + udd + \bar{s}d + \bar{d}u + (\bar{u}u - \bar{d}d)/\sqrt{2}$, so strangeness is conserved (so this isn't via the charged weak interactions). While the creation of the three $\bar{q}q$ pairs can take place via the EM or weak interactions, we need the strong interactions to form the hadrons (and break them up).

(k) $p \to e^+ + \gamma$: No, violates both baryon number and electron Lepton number.

(l) $p + p \to p + p + p + \overline{p}$: Yes, via the strong interactions[as in (j)].

(m) $n + \overline{n} \to \pi^+ + \pi^- + \pi^0$: Yes, by way of the strong interactions [see (j)].

(n) $\pi^+ + n \to \pi^- + p$: No, violates charge conservation.

(o) $K^- \to \pi^- + \pi^0$: Yes, via the weak interaction (we must change a strange quark to an up or down quark, so this requires flavor changing).

(p) $\Sigma^+ + n \to \Sigma^- + p$: No, violates charge conservation.

(q) $\Sigma^0 \to \Lambda + \gamma$: Yes, by way of the electromagnetic interaction (the quark content of the two hadrons is the same).

(r) $\Xi^- \to \Lambda + \pi^-$: Yes, via a combination of the charged weak interaction (to turn one of the strange quarks into a down quark) and either the strong (or to a lesser extent, electromagnetic or neutral weak) interaction to produce a $\bar{u}u$ pair (underlying process: $dss \to uds + \bar{u}d$).

(s) $\Xi^0 \to p + \pi^-$: Yes, by way of the weak interaction (the underlying interaction is $uss \to uud + \bar{u}d$). This interaction, however, will be highly suppressed, because it has to happen at "second order," or there are two flavor changing weak interactions (to get rid of the s quarks) that have to occur.

(t) $\pi^- + p \to \Lambda + K^0$: Yes, via the strong interaction: $\bar{u}d + uud \to uds + \bar{s}d$.

(u) $\pi^0 \to \gamma + \gamma$: Yes, via the electromagnetic interaction [see (b)].

(v) $\Sigma^- \to n + e + \bar{\nu}_e$: Yes, the underlying quark interaction is $dds \to udd$, and this will occur via the weak interaction ($s \to u$).

5. *Additional problem (Not something for you to turn in, but you'll be expected to understand this)*: What is the OZI rule, and how does it allow for the long lifetime of the J/ψ?

   This is explained in the text. Be sure you understand how this works, and why it's important.