Problem Set 3

Due: Sept. 17, 2008

1. In this problem, you will derive the Gell-Mann-Okubo mass formula for the octet baryons. You will need the transformation law for a 3 of $SU(3)$:

$$q_i \to U_{ij} q_j, \qquad \overline{q}_i \to \overline{q}_j (U^\dagger)_{ji}$$

where $U \in SU(3)$ and i, j are $SU(3)$ flavor indices.

(a) An octet of $SU(3)$ comes from $3 \otimes \overline{3} = 8 \oplus 1$, and two come from $3 \otimes 3 \otimes 3$. An antiquark and a specific combination of two quarks both transform in the $\overline{3}$ of $SU(3)$. Specifically,

$$\overline{q}_i \quad \text{and} \quad \epsilon_{ijk} q_j q_k$$

transform both as $\overline{3}$'s in $SU(3)$. Prove that these two objects transform the same.

(b) We can write a "nonet" combination of the quarks and antiquarks as

$$B_{ij} = q_i \overline{q}_j$$

where $i, j = 1, 2, 3$, and so B is a 3×3 matrix. B is a reducible representation of $3 \otimes \overline{3}$, as we know. Show how

$$\mathcal{B}_{ij} = B_{ij} - \frac{1}{3}\,\mathrm{Tr}(B)\delta_{ij}, \quad \text{and} \quad \mathrm{Tr}(B)$$

each transform under $SU(3)$ rotations. Note that we could now replace $\overline{q}_j$ with $\epsilon_{jik} q_i q_k$ to get the same transformation properties, and this would correspond to a baryon physically.

(c) Gell-Mann's assumption was that the strong interactions would contribute two terms to the masses of the baryons with the Hamiltonian:

$$H_{\text{strong}} = H_{\overline{m}} + H_{\delta m}$$

The first term would contribute the average mass, M_0, of a given multiplet (*ie*, if $SU(3)$ were exact we would just have

$$M_p = M_n = M_\Sigma = M_\Xi = M_\Lambda = M_0\ .$$

The second term splits the masses in a given multiplet. Since this is due to strong interactions it should be independent of charge and isospin (assuming that isospin is still a good symmetry), and thus must involve the $T^8 = \lambda^8/2$ generator of the octet, which is diagonal in isospin space. The two combinations

$$\mathrm{Tr}(\mathcal{B}^\dagger \mathcal{B} T^8), \text{ and } \mathrm{Tr}(\mathcal{B}^\dagger T^8 \mathcal{B})$$

are the only terms that break $SU(3)$ and are invariant under $SU(2)$ involving $\mathcal{B}$, so

$$H_{\delta m} = \delta m_1\,\mathrm{Tr}(\mathcal{B}^\dagger \mathcal{B} T^8) + \delta m_2\,\mathrm{Tr}(\mathcal{B}^\dagger T^8 \mathcal{B})$$

with two unknown parameters, $\delta m_{1,2}$.
We will use the following octet representation of the baryons:

$$\mathcal{B} = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$$

where all elements are complex operators, so $\mathcal{B}^\dagger$ contains $\Lambda^\dagger, p^\dagger$, etc., and the fact that a mass term for baryons are of the form

$$M_p p^\dagger p + M_n n^\dagger n + M_\Sigma \Sigma^\dagger \Sigma + \cdots$$

for baryons, we would like to determine the masses of the octet baryons. Evaluate the traces in $H_{\delta m}$ in terms of these baryons.

(d) For each baryon, add in the contribution M_0 from $H_{\overline{m}}$ (common to all baryons), and determine the masses of the baryons in terms of $M_0, \delta m_1,$ and δm_2. With isospin a good symmetry, we have

$$m_n = m_p \equiv m_N , \qquad m_{\Sigma^+} = m_{\Sigma^0} = m_{\Sigma^-} \equiv m_\Sigma , \qquad m_{\Xi^-} = m_{\Xi^0} \equiv m_\Xi$$

(e) Finally, use these equations to derive the Gell-Mann-Okubo relation

$$m_\Lambda = \frac{1}{3}\left[2(m_N + m_\Xi) - m_\Sigma\right]$$

2. Determine, using the quark model, the magnetic moments in units of the nuclear magneton, of all the members of the decuplet. Use whatever method you need to fix the unknown parameters (the constituent quark masses, m_u, m_d, m_s), for example you can use the quark model predictions for three of the octet baryons to set those, and then predict the decuplet. For those which have been measured, compare with the experimental number.

3. What about baryons with heavy quarks, the "charmed baryons" and "beautiful baryons"?

(a) Show that there are six spin-1/2 baryons containing a charm quark and two light quarks (u, d, s) that transform in the 6 of $SU(3)$, with flavor content

$$\Sigma_c^{++} \sim uuc\ , \quad \Sigma_c^{+} \sim udc\ , \quad \Sigma_c^{0} \sim ddc\ , \quad \Xi_c^{\prime +} \sim usc\ , \quad \Xi_c^{\prime 0} \sim dsc\ , \quad \Omega_c^{0} \sim ssc$$

and 3 baryons

$$\Xi_c^{+} \sim usc\ , \quad \Xi_c^{0} \sim dsc\ , \quad \Lambda_c^{+} \sim udc$$

which transform in the $\overline{3}$ of $SU(3)$. To do this, work out the different combinations that give a spin-1/2 baryon and use the fact that the total state must be completely symmetric in spin and flavor, since it is to combine with a completely antisymmetric state in color. Be sure to write down the full states, with the correct normalization, and describe the correct symmetry properties.

(b) Assuming (naively) the masses of baryons can be written as

$$m_B = m_1 + m_2 + m_3$$

where m_i are the "constituent quark masses," then we can estimate the masses of these charmed baryons. Given that the mass of the proton is roughly 1 GeV, we can approximate $m_u \approx m_d \approx 330$ MeV, and since the $\Omega^- \sim sss$ mass is roughly 1.7 GeV, $m_s \approx 570$ MeV. Use this to predict the masses of the above states. For this, you can assume the constituent charm quark mass is just its mass, 1.3 GeV.

(c) Look up in the particle data book the values of the masses of the above baryons (given that we are defining the parities to be positive, so these are all $1/2^+$ states), and compare those values to our naïve quark model estimates. [Note: You obviously won't be able to distinguish with this naïve estimate the differences between the sextet and anti-triplet.]

(d) Do the same as problem 3b but with $c \to b$, and $m_b = 5$ GeV. The only one of these baryons discovered is the $\Lambda_b^0 \sim udb$ (note the charge is different because the electric charge of the bottom quark is $-1/3e$), and it has a mass of $m_{\Lambda_b^0} = 5620.2$ MeV. Why should the quark model work so much better here? [What assumption did we implicitly use here to describe the baryon in terms of constituent quark masses?]

Additional Problem (not to be turned in)

1. When simulating fermions on a spacetime lattice, there is a duplication of quark species, so that for every quark flavor, you have 4 *tastes* (this is the so-called doubling problem). Usually one simulates three flavors, u, d, s, each with four tastes and the relevant symmetry group then is $SU(12)$.

(a) Using Young Tableaux, work out the number of mesons we would expect to see, and also the different dimensions of the baryons that would come out. In other words, work out $12 \otimes \overline{12}$ and $12 \otimes 12 \otimes 12$.

(b) Of the resulting irreducible representations that arise for baryons, which ones would correspond to "physical" multiplets [You can just make an argument here, no mathematical proof is needed.]?

(c) Now do the same with say, the charmed baryons. For heavy quarks, we find no doubling to occur (because of the heavy mass of the quark), so while the light quarks transform under $SU(12)$, the heavy quarks are $SU(12)$ singlets. What are the resulting charmed baryon states with a single charm quark (*ie*, what are the dimensions of the possible irreducible representations?)?