Problem Set 4

Due: Oct. 8, 2008

Note, for parts of this homework, you will have been expected to work through at least some of the derivation of the solutions to the Dirac Equation.

1. In the handout on the Dirac Equation, I defined the spin operator $\mathbf{S}$ defined in terms of $\mathbf{\Sigma}$. Prove that indeed the Dirac equation gives us a spin-1/2 particle. Do this using the Dirac Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$$

and show that

$$[H, \mathbf{L}] \neq 0$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is the orbital angular momentum operator. Determine the total angular momentum operator $\mathbf{J}$ in terms of $\mathbf{L}$ and $\mathbf{S}$, and show that this commutes with the Hamiltonian. Explain why this implies a spin of 1/2 for a Dirac field (in other words, show that $u^{(1),(2)}$ and $v^{(1),(2)}$ are eigenvectors of S^2, and give the eigenvalues).

2. Griffiths 7.43 [This is to calculate $e^- \mu^-$ scattering in the "Yukawa theory," where the photon is a massive spin-0 field, with a propagator given by $i/(q^2 - m_\gamma^2)$ and the 2-fermion-1-photon vertex is given by ie.]

3. Use crossing symmetry on your result from $|\mathcal{M}|^2$ in the previous problem to determine the prediction for $e^+ e^- \to \mu^+ \mu^-$ in the Yukawa Theory. Show that the spin-averaged cross section is given by what we showed in class in the limit $m_\gamma \to 0$.

4. The Lagrangian density for QED is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \overline{\psi}(i\gamma^\mu \partial_\mu + m)\psi + e\overline{\psi}\gamma^\mu A_\mu \psi$$

Use the definitions

$$\psi_L = \frac{1-\gamma_5}{2}\psi \ , \qquad \psi_R = \frac{1+\gamma_5}{2}\psi$$
$$\overline{\psi}_L = \overline{\psi}\frac{1+\gamma_5}{2} \ , \qquad \overline{\psi}_R = \overline{\psi}\frac{1-\gamma_5}{2}$$

to rewrite the Lagrangian in terms of the "left" and "right" handed fermions. Explain why this tells us that unless the mass of the fermion is zero, we do not have definite helicity states.