Problem Set 4

Due: Oct. 8, 2008

Note, for parts of this homework, you will have been expected to work through at least some of the derivation of the solutions to the Dirac Equation.

1. In the handout on the Dirac Equation, I defined the spin operator $\mathbf{S}$ defined in terms of $\mathbf{\Sigma}$. Prove that indeed the Dirac equation gives us a spin-1/2 particle. Do this using the Dirac Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$$

and show that

$$[H, \mathbf{L}] \neq 0$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is the orbital angular momentum operator. Determine the total angular momentum operator $\mathbf{J}$ in terms of $\mathbf{L}$ and $\mathbf{S}$, and show that this commutes with the Hamiltonian. Explain why this implies a spin of 1/2 for a Dirac field (in other words, show that $u^{(1),(2)}$ and $v^{(1),(2)}$ are eigenvectors of S^2, and give the eigenvalues).

Solution:

We can see that

$$\begin{aligned}
[H, L_i] &= \epsilon_{ijk}[\alpha_\ell p_\ell + \beta m, x_j p_k] \\
&= \epsilon_{ijk}[\alpha_\ell p_\ell, x_j p_k] + \epsilon_{ijk}[\beta m, x_j p_k] \\
&= \epsilon_{ijk}\alpha_\ell \{[p_\ell, x_j]p_k - x_j[p_\ell, p_k]\} \\
&= \epsilon_{ijk}\alpha_\ell - i\delta_{\ell j}p_k \\
&= -i\epsilon_{ijk}\alpha_j p_k = -i\boldsymbol{\alpha} \times \mathbf{p}
\end{aligned}$$

Thus, we can define the spin operator

$$S_i = \frac{1}{2}\Sigma_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

and with

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

we can see

$$\begin{aligned}
[H, S_i] &= \frac{1}{2}[\alpha_\ell p_\ell + \beta m, \Sigma_i] \\
&= \frac{1}{2}[\alpha_\ell, \Sigma_i]p_\ell \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma_\ell \\ \sigma_\ell & 0 \end{pmatrix}, \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\right] p_\ell \\
&= \frac{1}{2}\begin{pmatrix} 0 & [\sigma_\ell, \sigma_i] \\ [\sigma_\ell, \sigma_i] & 0 \end{pmatrix} p_\ell \\
&= -i\epsilon_{j\ell k}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} p_\ell \\
&= -i\epsilon_{j\ell k}\alpha_k p_\ell \\
&= +i\boldsymbol{\alpha} \times \mathbf{p}
\end{aligned}$$

and so

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

is conserved. Acting S on all of the spinors we get $3/4 = s(s+1)$, implying the spin of the Dirac fermion is 1/2. Note that while acting S_z on the spinors gives $\pm 1/2$, this is not completely sufficient to show that the total spin is 1/2, as a spin-3/2 particle has S_z components of $\pm 1/2$.

2. Griffiths 7.43 [This is to calculate $e^- \mu^-$ scattering in the "Yukawa theory," where the photon is a massive spin-0 field, with a propagator given by $i/(q^2 - m_\gamma^2)$ and the 2-fermion-1-photon vertex is given by ie.]

Solution:

(a) There is only one diagram, and we assign $p(p')$ to the incoming (outgoing) electron, and we assign $k(k')$ to the incoming (outgoing) muon. $q = p - p' = k - k'$ is the momentum of the intermediate scalar field. All external lines are particles, so we only have $u, \bar{u}$'s.

$$-i\mathcal{M} = \bar{u}^{(s')}(p')(ie)u^{(s)}(p)\frac{i}{q^2 - m_\gamma^2}\bar{u}^{(r')}(k')(ie)u^{(r)}(k)$$

or

$$\mathcal{M} = e^2 \frac{\bar{u}^{(s')}(p')u^{(s)}(p)\bar{u}^{(r')}(k')u^{(r)}(k)}{q^2 - m_\gamma^2}$$

(b)

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{ss'rr'}\frac{e^4}{(q^2 - m_\gamma^2)^2}\bar{u}^{(s')}(p')u^{(s)}(p)\bar{u}^{(s)}(p)u^{(s')}(p')\bar{u}^{(r')}(k')u^{(r)}(k)\bar{u}^{(r)}(k)u^{(r')}(k') \\
&= \frac{1}{4}\frac{e^4}{(q^2 - m_\gamma^2)^2}\,\mathrm{Tr}[(\not{p}' + m_e)(\not{p} + m_e)]\,\mathrm{Tr}[(\not{k}' + m_\mu)(\not{k} + m_\mu)]
\end{aligned}$$

(c) Neglecting the masses:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\frac{e^4}{(q^2 - m_\gamma^2)^2}\,\mathrm{Tr}[\not{p}'\not{p}]\,\mathrm{Tr}[\not{k}'\not{k}] \\
&= \frac{4e^4}{(q^2 - m_\gamma^2)^2}(p \cdot p')(k \cdot k')
\end{aligned}$$

We have $S = 1$ in the expression for the cross section, and we have the energies of all particles the same, E. Also we will set for the four-momenta

$$p = (E, E\hat{z}), \quad p' = (E, E\hat{k}), \quad k = (E, -E\hat{z}), \quad k' = (E, -E\hat{k})$$

with $\hat{k} \cdot \hat{z} = \cos\theta$. Also, $q^2 = (p' - p)^2 = -2p \cdot p'$. So:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{CM} &= \frac{1}{64\pi^2(2E)^2}\langle|\mathcal{M}|^2\rangle \\
&= \frac{1}{16(64\pi^2E^2)}\frac{e^4}{(q^2 - m_\gamma^2)^2}\,\mathrm{Tr}[\not{p}'\not{p}]\,\mathrm{Tr}[\not{k}'\not{k}] \\
&= 16\frac{1}{16(64\pi^2E^2)}\frac{e^4}{(q^2 - m_\gamma^2)^2}(p' \cdot p)(k' \cdot k) \\
&= \frac{1}{64\pi^2E^2}\frac{e^4}{(q^2 - m_\gamma^2)^2}(p' \cdot p)(k' \cdot k) \\
&= \frac{1}{64\pi^2E^2}\frac{e^4}{(q^2 - m_\gamma^2)^2}(E^2 - E^2\cos\theta)^2 \\
&= \frac{\alpha^2}{4E^2}\frac{1}{[2(1 - \cos\theta) + (m_\gamma/E)^2]^2}(1 - \cos\theta)^2
\end{aligned}$$

(d) Assuming $m_\gamma \gg E$, we can neglect the $1-\cos\theta$ term:

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{CM} &= \frac{\alpha^2}{4E^2}\frac{E^4}{m_\gamma^4}(1-\cos\theta)^2 \\ \sigma &= \frac{4\pi\alpha^2 E^2}{3m_\gamma^4} \end{aligned}$$

(e) So in the lab frame, in the limit $|\mathbf{p}_e| \ll m_e \ll m_\gamma \ll m_\mu$, we can write the energy of the electron as:

$$E = \sqrt{|\mathbf{p}_e|^2 + m_e^2} \approx m_e\left(1 + \frac{1}{2}\frac{|\mathbf{p}_e|^2}{m_e^2}\right)$$

and we have

$$p = (E, |\mathbf{p}_e|\hat{z}), \quad p' = (E, |\mathbf{p}_e|\hat{k}), \quad k = (m_\mu, 0), \quad k' = (m_\mu, 0)$$

Again $\hat{k}\cdot\hat{z} = \cos\theta$, and we are neglecting the final three-momentum of the muon, as we assume it does not recoil. Also, in this case

$$q^2 = (p'-p)^2 = -2|\mathbf{p}_e|^2(1-\cos\theta) = -4|\mathbf{p}_e|^2\sin^2(\theta/2)$$

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\frac{e^4}{(q^2-m_\gamma^2)^2}\,\mathrm{Tr}[\not{p}'\not{p}]\,\mathrm{Tr}[(\not{k}'+m_\mu)(\not{k}+m_\mu)] \\ &= \frac{4e^4}{(q^2-m_\gamma^2)^2}(p\cdot p')(k\cdot k + m_\mu^2) \\ &= \frac{8e^4}{[4|\mathbf{p}_e|^2\sin^2(\theta/2)+m_\gamma^2]^2}(E^2 - |\mathbf{p}_e|^2\cos\theta)m_\mu^2 \\ &= \frac{16e^4}{[4|\mathbf{p}_e|^2\sin^2(\theta/2)+m_\gamma^2]^2}(E^2 - |\mathbf{p}_e|^2\cos\theta)m_\mu^2 \end{aligned}$$

and now, with the result of Problem 6.10, using the fact that $|\mathbf{p}_e|^2 \gg m_\gamma^2$

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} &= \frac{E^2}{64\pi^2 m_\mu^2 E^2}\langle|\mathcal{M}|^2\rangle \\ &= \frac{4\alpha^2}{[4|\mathbf{p}_e|^2\sin^2(\theta/2)+m_\gamma^2]^2}[m_e^2 + |\mathbf{p}_e|^2(1-\cos\theta)] \\ &\approx \frac{\alpha^2}{4|\mathbf{p}_e|^4\sin^4(\theta/2)}m_e^2 \\ &= \left(\frac{\alpha m_e}{2|\mathbf{p}_e|^2\sin^2(\theta/2)}\right)^2 \\ &= \left(\frac{\alpha}{2m_e v^2\sin^2(\theta/2)}\right)^2 \end{aligned}$$

This is precisely the Rutherford cross-section! Note it doesn't matter what type of particle is exchanged to obtain this cross-section, and this is due to the non-relativistic limit. Obviously the total cross-section is infinite.

3. Use crossing symmetry on your result from $|\mathcal{M}|^2$ in the previous problem to determine the prediction for $e^+e^- \to \mu^+\mu^-$ in the Yukawa Theory. Show that the spin-averaged cross section is given by what we showed in class in the limit $m_\gamma \to 0$.

Solution:

The amplitude for

$$e^-(p)e^+(p') \to \mu^+(k)\mu^-(k')$$

is the same as that for

$$e^-(p)\mu^-(-k) \to e^-(-p')\mu^-(k')$$

and the four-momenta are as above.

The amplitude above would be the same as the amplitude for So we have, with $q^2 = (p+p')^2$, and setting $m_\gamma = 0$:

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{4e^4}{(q^2)^2}(p\cdot p' - m_e^2)(k\cdot k' - m_\mu^2) \\
&= \frac{e^4}{4E^4}(2E^2 - 2m_e^2)(2E^2 - 2m_\mu^2) \\
&= e^4\left(1 - \frac{m_e^2}{E^2}\right)\left(1 - \frac{m_\mu^2}{E^2}\right)
\end{aligned}$$

so

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{CM} &= \frac{1}{64\pi^2(2E)^2}\frac{|\mathbf{p}_\mu|}{|\mathbf{p}_e|}\langle|\mathcal{M}|^2\rangle \\
&= \frac{\alpha^2}{16E^2}\left(1 - \frac{m_e^2}{E^2}\right)^{1/2}\left(1 - \frac{m_\mu^2}{E^2}\right)^{3/2} \\
\sigma &= \frac{\pi\alpha^2}{4E^2}\left(1 - \frac{m_e^2}{E^2}\right)^{1/2}\left(1 - \frac{m_\mu^2}{E^2}\right)^{3/2}
\end{aligned}$$

which is precisely what I showed in class.

4. The Lagrangian density for QED is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\psi}(i\gamma^\mu\partial_\mu + m)\psi + e\overline{\psi}\gamma^\mu A_\mu\psi$$

Use the definitions

$$\begin{aligned}
\psi_L &= \frac{1-\gamma_5}{2}\psi\ , \qquad \psi_R = \frac{1+\gamma_5}{2}\psi \\
\overline{\psi}_L &= \overline{\psi}\frac{1+\gamma_5}{2}\ , \qquad \overline{\psi}_R = \overline{\psi}\frac{1-\gamma_5}{2}
\end{aligned}$$

to rewrite the Lagrangian in terms of the "left" and "right" handed fermions. Explain why this tells us that unless the mass of the fermion is zero, we do not have definite helicity states.

Solution:

In this case, we can use the fact that the $(1 \pm \gamma_5)/2$ are projectors

$$P_L = \frac{1-\gamma_5}{2}, \quad P_R = \frac{1+\gamma_5}{2}$$

satisfy $P_L^2 = P_L, P_R^2 = P_R, P_L + P_R = 1$, so

$$\begin{aligned}
\psi &= \psi_L + \psi_R \\
\overline{\psi} &= \overline{\psi}_L + \overline{\psi}_R
\end{aligned}$$

So we have

$$\overline{\psi}\psi = (\overline{\psi}_L + \overline{\psi}_R)(\psi_L + \psi_R)$$

and the "diagonal" terms vanish:

$$\overline{\psi}_L\psi_L = \overline{\psi}P_R P_L\psi = 0$$

so

$$\overline{\psi}\psi = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L$$

For the terms with the γ^μ, we use the fact that

$$P_L\gamma^\mu = \gamma^\mu P_R$$

because γ_5 anticommutes with all gamma matrices. In this case, the "diagonal" terms then don't vanish, but the cross terms do, so

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi}_L\gamma^\mu\psi_L + \overline{\psi}_R\gamma^\mu\psi_R$$

and we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\psi}_L(i\gamma^\mu\partial_\mu)\psi_L + \overline{\psi}_R(i\gamma^\mu\partial_\mu)\psi_R + m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R) + e\overline{\psi}_L\gamma^\mu A_\mu\psi_L + e\overline{\psi}_R\gamma^\mu A_\mu\psi_R$$

So the mass term is a two-point interaction between the left- and right-handed fermions. If this mass is set to zero, then the left- and right-handed fermions do not interact at all. This is called a "chiral symmetry."