

Problem Set 5 Solutions

Due: Oct. 20, 2007

Problems 1-3 concern an effective theory for pions and nucleons, where m_π is the common mass of all three pions and M is the mass of the nucleons. The interaction Lagrangian is given by

$$\mathcal{L}_I = -ig_A \left[\sqrt{2} \bar{\psi}_p \gamma_5 \psi_n \pi^+ + \sqrt{2} \bar{\psi}_n \gamma_5 \psi_p \pi^- + (\bar{\psi}_p \gamma_5 \psi_p - \bar{\psi}_n \gamma_5 \psi_n) \pi^0 \right]$$

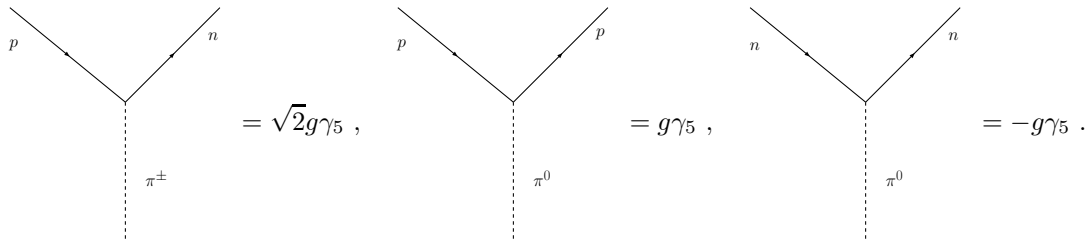
and there is only one coupling, restricted by isospin invariance.

1. For \mathcal{L}_I to be Lorentz invariant, how must the pions transform under Lorentz transformations (scalar, pseudoscalar, vector, axial-vector, or tensor)?

Solution:

We know the object $\bar{\psi}\psi$ is a Lorentz scalar, and $\bar{\psi}\gamma_5\psi$ is a Lorentz pseudoscalar. Thus, all three pions must be pseudoscalars under Lorentz transformations to ensure a Lorentz-invariant Lagrangian.

2. The interaction vertices are given by



There are 10 possible pion-nucleon scattering events:

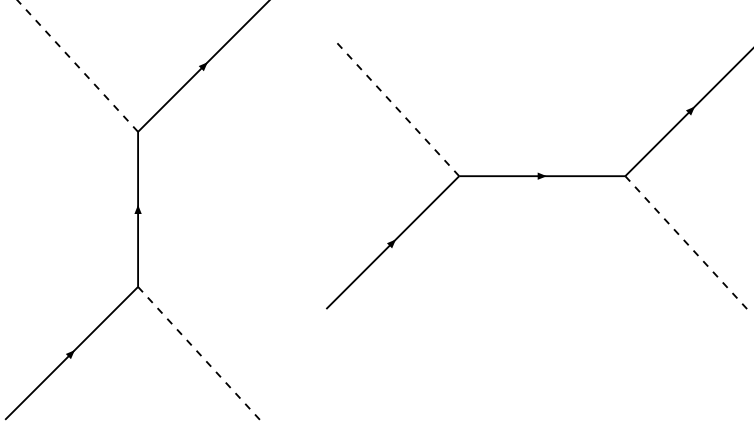
- (a) $\pi^+ + p \rightarrow \pi^+ + p$
- (b) $\pi^0 + p \rightarrow \pi^0 + p$
- (c) $\pi^- + p \rightarrow \pi^- + p$
- (d) $\pi^+ + n \rightarrow \pi^+ + n$
- (e) $\pi^0 + n \rightarrow \pi^0 + n$
- (f) $\pi^- + n \rightarrow \pi^- + n$
- (g) $\pi^+ + n \rightarrow \pi^0 + p$
- (h) $\pi^0 + n \rightarrow \pi^- + p$
- (i) $\pi^- + p \rightarrow \pi^0 + n$
- (j) $\pi^0 + p \rightarrow \pi^+ + n$

Calculate the amplitudes for these processes, and use isospin to relate them to just two amplitudes: One with isospin 1/2 and one with isospin 3/2: $\mathcal{M}_{1/2}, \mathcal{M}_{3/2}$.

Solution:

We will use a different solution method than writing out the different isospins, as that can be seen in the text.

All diagrams here will have an intermediate fermion, and either be an s -channel or a t -channel diagram, or both. We have three classes of diagrams. They are one or both of the following:



Processes a, b, e, and f use just the left-hand diagram, processes c and d involve the right-hand diagram, and the charge-exchange processes (g,h,i,j) need both diagrams added together.

Since we are setting all pion masses equal and the nucleon masses the same, we can just write down the diagrams very easily. The only difference will be with additional minus signs or factors of $\sqrt{2}$, coming from the vertices. We'll put those in afterwards. We will assign momenta as follows:

$$\begin{aligned} \text{incoming } \pi : p & & \text{outgoing } \pi : p' \\ \text{incoming nucleon : } k & & \text{outgoing nucleon : } k' \end{aligned}$$

First, the left-hand amplitude is (we'll put the vertex factors of $g, \sqrt{2}g$, or $-g$ in the end when we put these together for each process), and suppressing spin indices:

$$i\mathcal{M}_L = \frac{\bar{u}(k')\gamma_5(\not{k} - \not{p}' + M)\gamma_5 u(k)}{t - M^2}$$

with $t = (p' - k)^2$, and the right-hand amplitude

$$i\mathcal{M}_R = \frac{\bar{u}(k')\gamma_5(\not{p} + \not{k} + M)\gamma_5 u(k)}{s - M^2}$$

Both of these can be simplified, using

$$\not{k}\gamma_5 u(k) = -M\gamma_5 u(k)$$

so

$$\begin{aligned} i\mathcal{M}_L &= -\frac{\bar{u}(k')\not{p}'u(k)}{t - M^2} \\ i\mathcal{M}_R &= -\frac{\bar{u}(k')\not{p}u(k)}{s - M^2} \end{aligned}$$

And with this, we have the following sets of amplitudes:

$$\begin{aligned}
i\mathcal{M}_a &= 2g^2\mathcal{M}_L \\
i\mathcal{M}_b &= g^2\mathcal{M}_L \\
i\mathcal{M}_c &= 2g^2\mathcal{M}_R \\
i\mathcal{M}_d &= 2g^2\mathcal{M}_R \\
i\mathcal{M}_e &= g^2\mathcal{M}_L \\
i\mathcal{M}_f &= 2g^2\mathcal{M}_L \\
i\mathcal{M}_g &= \sqrt{2}g^2[\mathcal{M}_R - \mathcal{M}_L] \\
i\mathcal{M}_h &= \sqrt{2}g^2[-\mathcal{M}_R + \mathcal{M}_L] \\
i\mathcal{M}_i &= \sqrt{2}g^2[-\mathcal{M}_R + \mathcal{M}_L] \\
i\mathcal{M}_j &= \sqrt{2}g^2[\mathcal{M}_R - \mathcal{M}_L]
\end{aligned}$$

This is simpler since we can calculate these explicitly based on the diagrams. See Griffiths for the deconstruction of these in terms of isospin amplitudes.

- Calculate the total cross sections for elastic π^+p and π^-n scattering in the center of mass frame. These cross sections do not hold near $E_{CM} \approx 1232$ MeV. Explain.

Solution:

These two amplitudes are (a) and (f) from above, and clearly they are equal. We will average over the initial nucleon spin and sum over the final spin, and we get

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{2} \sum \frac{4g^4}{(t - M^2)^2} \bar{u}^{(s')}(k') \not{p}' u^{(s)}(k) \bar{u}^{(s)}(k) \not{p}' u^{(s')}(k') \\
&= \frac{2g^4}{(t - M^2)^2} \text{Tr}[(\not{k}' + M)\not{p}'(\not{k} + M)\not{p}'] \\
&= \frac{2g^4}{(t - M^2)^2} \text{Tr}[\not{k}'\not{p}'\not{k}\not{p}' + M^2(p')^2] \\
&= \frac{2g^4}{(t - M^2)^2} 4[2(k' \cdot p')(k \cdot p') - (k' \cdot k)m_\pi^2 + M^2m_\pi^2] \\
&= \frac{8g^4}{(t - M^2)^2} [2(k' \cdot p')(k \cdot p') - (k' \cdot k)m_\pi^2 + M^2m_\pi^2]
\end{aligned}$$

Let's simplify this some in the CM frame. Also, given the fact that all the masses are squared, and

$$\frac{m_\pi^2}{M^2} \approx 0.01$$

we will neglect the pion mass. Here we have

$$p = (E, \mathbf{p}), \quad k = (E_N, -\mathbf{p})$$

$$p' = (E, \mathbf{k}), \quad k' = (E_N, -\mathbf{k})$$

With $\mathbf{p} \cdot \mathbf{k} = E^2 \cos \theta$, where $E = |\mathbf{p}| = |\mathbf{k}|$ is a 3-momentum magnitude. So we have

$$p' \cdot k' = p \cdot k = E(E_N + E), \quad p \cdot k' = p' \cdot k = E(E_N + E \cos \theta), \quad k \cdot k' = E_N^2 - E^2 \cos \theta$$

$$t = (p' - p)^2 = -2E^2(1 - \cos \theta)$$

We can also write

$$E_N = \sqrt{p^2 + M^2} = \sqrt{E^2 + M^2}$$

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{16g^4}{(2E^2(1 - \cos \theta) + M^2)^2} E^2 (E_N + E)(E_N + E \cos \theta) \\
&= \frac{16g^4 E^4}{(2E^2(1 - \cos \theta) + M^2)^2} \left(1 + \sqrt{1 + \frac{M^2}{E^2}} \right) \left(\sqrt{1 + \frac{M^2}{E^2}} + \cos \theta \right) \\
&= \frac{16g^4}{(2(1 - \cos \theta) + M^2/E^2)^2} \left(1 + \sqrt{1 + \frac{M^2}{E^2}} \right) \left(\sqrt{1 + \frac{M^2}{E^2}} + \cos \theta \right)
\end{aligned}$$

So the differential cross-section is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{16g^4}{64\pi^2} \frac{1}{(E + E_N)^2 (2(1 - \cos \theta) + M^2/E^2)^2} \left(1 + \sqrt{1 + \frac{M^2}{E^2}} \right) \left(\sqrt{1 + \frac{M^2}{E^2}} + \cos \theta \right) \\
&= \frac{g^4}{4\pi^2 E^2} \frac{1}{\left(1 + \sqrt{1 + \frac{M^2}{E^2}} \right)^2 (2(1 - \cos \theta) + M^2/E^2)^2} \left(1 + \sqrt{1 + \frac{M^2}{E^2}} \right) \left(\sqrt{1 + \frac{M^2}{E^2}} + \cos \theta \right)
\end{aligned}$$

Let's not bother with other simplifications now, and just explicitly evaluate the integral. We have

$$\begin{aligned}
\int_0^\pi \frac{1}{(2(1 - \cos \theta) + M^2/E^2)^2} \sin \theta d\theta &= \frac{2E^4}{M^2(4E^2 + M^2)} \\
\int_0^\pi \frac{\cos \theta}{(2(1 - \cos \theta) + M^2/E^2)^2} \sin \theta d\theta &= \frac{E^2}{4M^2} + \frac{E^2}{2(4E^2 + M^2)} - \frac{1}{4} \ln \left[1 + \frac{4E^2}{M^2} \right]
\end{aligned}$$

so

$$\begin{aligned}
\sigma &= \frac{g^4}{2\pi E^2} \frac{1}{1 + \sqrt{1 + \frac{M^2}{E^2}}} \left(\sqrt{1 + \frac{M^2}{E^2}} \frac{2E^4}{M^2(4E^2 + M^2)} + \frac{E^2}{4M^2} \right. \\
&\quad \left. + \frac{E^2}{2(4E^2 + M^2)} - \frac{1}{4} \ln \left[1 + \frac{4E^2}{M^2} \right] \right)
\end{aligned}$$

This is the same for both processes, and in fact there was an error in the problem. This cross-section doesn't hold near 1232 MeV, because of the fact that the Δ would contribute. But this contribution would be the same for both of these, because the processes have the same isospin. For a process which takes contributions from different isospin components (see Griffiths), then contributions from the Δ would have different forms. This was what the problem was supposed to address, but I had a typo that I didn't notice when asking you to calculate these diagrams.

Note, as expected, if we send $M \rightarrow 0$, this cross-section goes to ∞ .

4. Show that the scattering amplitude and the total cross section are related by the identity

$$\sigma = \frac{4\pi}{k} \text{Im} f_{\mathbf{k}}(0)$$

which is known as the *Optical Theorem*.

Solution:

Given

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_\ell$$

and

$$f_{\mathbf{k}}(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin \delta_\ell e^{i\delta_\ell} P_\ell(\cos \theta)$$

We can take the imaginary part of the scattering amplitude

$$\text{Im}f_k(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_{\ell} P_{\ell}(\cos \theta)$$

and set $\theta = 0$ (this corresponds to forward scattering), and we can use the fact that the Legendre polynomials are normalized to 1 at this value of $\cos \theta$, regardless of ℓ :

$$\text{Im}f_k(0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_{\ell}$$

From here, what we wished to prove is straightforward.

5. In connecting the relativistic normalization used in field theory, the first Born approximation reads

$$\langle p' | iT | p \rangle = -i\tilde{V}(\mathbf{p}' - \mathbf{p})(2\pi)\delta(E_{\mathbf{p}'} - E_{\mathbf{p}})$$

where \tilde{V} is the Fourier transform of the potential, and T is the Transition matrix which we relate to the amplitude. Show that in the non-relativistic limit, $e^+e^- \rightarrow e^+e^-$ scattering in QED does indeed give the Coulomb potential, while in the Yukawa theory (replacing the photon with a massive spin-0 boson), you get

$$V(r) \propto \frac{e^{-mr}}{r}$$

Also show that the Yukawa theory is *always* attractive, while for QED it depends on the sign of the fermions. [Note, this is why Yukawa proposed this for the strong interactions (see the pion-nucleon theory above), as it ensured a purely attractive theory for the nucleons.]

Solution:

Before we actually get through working this problem, we note that since we are interested in the scattering amplitude itself, we have to be careful with the ordering of the fermions in the matrix element. So given a scattering process of $e^+e^- \rightarrow e^+e^-$, we want to evaluate the matrix element

$$\langle \mathbf{k}', \mathbf{p}' | e^{-iH_{\text{int}}t} | \mathbf{k}, \mathbf{p} \rangle$$

with the p 's the electron momenta and the k 's the positron momenta. Now, we would expand the interacting Hamiltonian to quadratic order in e to get the terms that allow electron-positron scattering. The photon fields (or scalar fields for the Yukawa theory) will Wick contract to get our propagator, but what we are interested in are the fermions. Schematically, this looks like

$$\langle \mathbf{k}', \mathbf{p}' | \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 | \mathbf{k}, \mathbf{p} \rangle$$

For e^+e^- scattering, the external states are created by:

$$\langle 0 | \bar{\psi}_{\mathbf{k}'} \psi_{\mathbf{p}'} \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \psi_{\mathbf{k}} \bar{\psi}_{\mathbf{p}} | 0 \rangle$$

Performing the contractions, and moving around the fields, we get the following (where we must have the $\psi\bar{\psi}$ ordering, and square brackets mean the fields are contracted):

$$\langle 0 | \bar{\psi}_{\mathbf{k}'} \psi_{\mathbf{p}'} \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \psi_{\mathbf{k}} \bar{\psi}_{\mathbf{p}} | 0 \rangle = -\langle 0 | [\psi_{\mathbf{p}'} \bar{\psi}_1] [\psi_1 \bar{\psi}_{\mathbf{k}'}] [\psi_{\mathbf{k}} \bar{\psi}_2] [\psi_2 \bar{\psi}_{\mathbf{p}}] | 0 \rangle$$

(there is another contraction with the same sign, and that is switching $1 \leftrightarrow 2$). Whereas if we have e^-e^- scattering:

$$\langle 0 | \psi_{\mathbf{k}'} \psi_{\mathbf{p}'} \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \bar{\psi}_{\mathbf{k}} \bar{\psi}_{\mathbf{p}} | 0 \rangle = +\langle 0 | [\psi_{\mathbf{k}'} \bar{\psi}_1] [\psi_{\mathbf{p}'} \bar{\psi}_2] [\psi_1 \bar{\psi}_{\mathbf{p}}] [\psi_2 \bar{\psi}_{\mathbf{k}}] | 0 \rangle$$

The same result occurs for e^+e^+ scattering. So in addition to the actual matrix elements we will evaluate, there is an additional minus sign for the oppositely charged particles.

For QED and e^+e^- scattering, we have

$$i\mathcal{M} = -\frac{ie^2}{q^2} \bar{u}^{(s')}(p') \gamma_\mu u^{(r')}(k') \bar{v}^{(s)}(k) \gamma^\mu v^{(r)}(p)$$

The initial minus sign comes from the analysis above. Using the Dirac representation for the gamma matrices, it is explicit that in the NR limit

$$\gamma^0 u = \gamma^0 u, \quad \gamma^i u = 0$$

and the same for the v 's. So:

$$i\mathcal{M} = -\frac{ie^2}{q^2} (u^{(s')})^\dagger(p') u^{(r')}(k') (v^{(s)})^\dagger(k) v^{(r)}(p)$$

Also in this limit, we assume very little 3-momentum transfer, so we can use the normalization of the spinors, true only when the spin is the same for both:

$$u^\dagger(p') u(k') = v^\dagger(k) v(p) = 2m$$

(Note this is really $2E_{\mathbf{p}}$, but the three-momentum is being neglected.) So we then have

$$i\mathcal{M} = -\frac{ie^2}{q^2} (2m\delta_{s'r'}) (2m\delta_{sr})$$

Also, we have

$$q^2 = (p - p')^2 \approx -|\mathbf{p} - \mathbf{p}'|^2 + \dots$$

in the NR limit, and

$$i\mathcal{M} = \frac{ie^2}{|\mathbf{q}|^2} (2m\delta_{s'r'}) (2m\delta_{sr})$$

The $2m$'s come from our relativistic normalization, so we drop them in comparing to the Born approximation, and we have

$$\tilde{V}(\mathbf{q}) = \frac{e^2}{|\mathbf{q}|^2}$$

To do this Fourier transform, we have to be careful, so we add in a small mass, and then we can perform the integral by contour integration, closing the contour in the upper half plane:

$$\begin{aligned} V(\mathbf{x}) &= -e^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{|\mathbf{q}|^2 + m^2} \\ &= -\frac{e^2}{8\pi^3} \int_0^\infty q^2 dq \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \frac{e^{iqx \cos\theta}}{q^2 + m^2} \\ &= -\frac{e^2}{4\pi^2} \int_0^\infty q^2 dq \int_{-1}^1 dy \frac{e^{iqxy}}{q^2 + m^2} \\ &= -\frac{e^2}{4\pi^2} \int_0^\infty q^2 dq \frac{e^{iqx} - e^{-iqx}}{iqx} \frac{1}{q^2 + m^2} \\ &= i \frac{e^2}{4\pi^2 |\mathbf{x}|} \int_0^\infty q dq \frac{e^{iqx} - e^{-iqx}}{q^2 + m^2} \\ &= i \frac{e^2}{4\pi^2 |\mathbf{x}|} \int_{-\infty}^\infty q dq \frac{e^{iqx}}{q^2 + m^2} \\ &= -\frac{e^2}{4\pi} \frac{e^{-mx}}{|\mathbf{x}|} \end{aligned}$$

which is an attractive potential, and for this to be QED, we take $m \rightarrow 0$:

$$V(\mathbf{x}) = -\frac{\alpha}{|\mathbf{x}|}$$

If we were to switch to like-charged particles scattering, then we wouldn't get the minus sign coming from the Wick contractions, and we would change one of the sets of u 's to v 's or vice versa, which gives the same sign from the normalization, and ultimately we would get

$$V(\mathbf{x}) = +\frac{\alpha}{|\mathbf{x}|}$$

for e^+e^+ or e^-e^- scattering, which is repulsive.

For the Yukawa theory, most of the steps follow, and the only difference is we don't get $u^\dagger u$, but $\bar{u}u$ normalizations, and these are

$$\bar{u}(p')u(k') = -\bar{v}(k)v(p) = 2m$$

Other than this all the steps follow from above, and we can see that for oppositely charged particles, we get an overall sign from the Wick contractions and an overall sign from the opposite norm's, so there is no overall sign. For like-charged particles, we get no sign for the Wick contractions, and no sign for the norms (because we'll have $\bar{u}u\bar{u}u$ or $\bar{v}v\bar{v}v$, so the signs in the latter term cancel). Thus again, no overall sign, like the e^+e^- case for QED.

So we can just switch $e \rightarrow g$, the coupling for the Yukawa theory and we get

$$V(\mathbf{x}) = -\frac{g^2}{4\pi} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|}$$

where we won't send the mass to zero here, since the Yukawa theory officially has a massive exchange boson. This is the potential for all cases, and such is universally attractive potential.