Problem Set 6 ("Midterm") Solutions

Due: By classtime, Oct. 27, 2008, **no exceptions**

For this assignment, you are not allowed to work together, although you can use any notes and texts. Additionally, if you have any questions on the problems, email me (as I will be out of town this is your only option), and I will respond to all students with my answer. Overall this assignment will be part of your homework grade, but is worth **100 points**.

1. [10 pts] Derive the full vertex in terms of the electromagnetic form factor, $G_\pi(q^2)$ for the π^+, recalling that the pions are spin-zero.

   Solution:

   Here the full vertex will be of the form
   $$\langle p'|\Gamma^\mu|p\rangle$$
   and the pions have no spin. We only have two four-vectors in this problem, $p^\mu, (p')^\mu$, and they can come in two ways:
   $$\langle p'|\Gamma^\mu|p\rangle = G_\pi(q^2)(p+p')^\mu + G'_\pi(q^2)(p'-p)^\mu$$
   The first term is precisely what the tree-level vertex is in Scalar QED. The second term must vanish because of charge conservation, since dotting $q = p' - p$ into this gives
   $$q_\mu\langle p'|\Gamma^\mu|p\rangle = G'_\pi(q^2)q^2$$
   and this must vanish. So we have
   $$\langle p'|\Gamma^\mu|p\rangle = G_\pi(q^2)(p+p')^\mu$$

2. [10 pts] Show using charge conjugation invariance that the electromagnetic form factor for the π^0 is zero (use the same result from problem 1 to get the form, and then show that $G_\pi = 0$ for all q^2 with charge conjugation invariance). Why does this not hold for the neutron?

   Solution:

   No we can start from the previous problem, which would hold for any spin-zero particle, including the π^0. We can show this vanishes though, by looking at the full coupling to a photon:
   $$\langle p'|\Gamma^\mu|p\rangle A_\mu$$
   As we saw in class, the π^0 is a $C = +1$ state, while the photon is a $C = -1$ state. Thus, under a C transformation, we have
   $$\langle p'|\Gamma^\mu|p\rangle A_\mu \rightarrow_C -\langle p'|\Gamma^\mu|p\rangle A_\mu$$
   Since C-invariance is a symmetry of QED, these two must be equal, and thus $G_\pi(q^2) = 0$ for all q^2 for the neutral pion.

   This doesn't hold for the neutron because the neutron is not its own antiparticle. Thus, while the F_1 form factor for the neutron vanishes at $q^2 = 0$, due to its vanishing charge, it does *not* vanish *a priori* for non-zero q^2.

3. [30 pts] Now derive the electromagnetic form factors for the ρ^+ meson. The full vertex will have the form

$$\epsilon^*_\alpha(p')\left(e\Gamma^{\alpha\mu\beta}\right)\epsilon_\beta(p)$$

where the polarization tensors of the ρ satisfy

$$p'^{\alpha}\epsilon_\alpha(p') = p^\beta\epsilon_\beta(p) = 0$$

and

$$p^2 = (p')^2 = m_\rho^2$$

Be sure to write down *all possible* terms that arise due to Lorentz invariance, and use charge conservation ($(p'-p)_\mu\Gamma^{\alpha\mu\beta} = 0$) to reduce the number of form factors to three. [In general, for a particle of spin-s, there are $2s+1$ electromagnetic form-factors which in the $q^2 \to 0$ limit, lead to $2s+1$ electromagnetic moments. In the case of a spin-one field, you get in addition to the electric charge and magnetic dipole moment, the electric quadrupole moment.]

Solution:

This is a more involved problem. We have to form a rank-3 tensor, so we need three indices. We have the following objects to combine:

$$g^{\mu\nu}, q^\mu, (p+p')^\mu$$

where $q = p' - p$ as usual. So we have the following possible terms, where all the functions $A, B, \ldots$ are functions only of q^2 as usual:

$$\begin{aligned}\Gamma^{\alpha\mu\beta} \;=\;& Ag^{\alpha\mu}q^\beta + Bg^{\alpha\beta}q^\mu + Cg^{\beta\mu}q^\alpha + Dg^{\alpha\mu}(p+p')^\beta + Eg^{\alpha\beta}(p+p')^\mu + Fg^{\beta\mu}(p+p')^\alpha \\ &+ Gq^\alpha q^\mu q^\beta + H(p+p')^\alpha(p+p')^\beta q^\mu + I(p+p')^\mu(p+p')^\beta q^\alpha + J(p+p')^\alpha(p+p')^\mu q^\beta \\ &+ K(p+p')^\alpha(p+p')^\beta(p+p')^\mu + Lq^\alpha q^\beta(p+p')^\mu + Mq^\alpha q^\mu(p+p')^\beta + Nq^\mu q^\beta(p+p')^\alpha\end{aligned}$$

Phew. Okay, so now we want to dot q_μ into this, and all terms that have $(p+p')^\mu$ in them vanish automatically, but any terms with q^μ won't, so this means we must set $B = G = H = M = N = 0$. There are more relations, so let's set these to zero and then dot q into the remaining:

$$q_\mu\Gamma^{\alpha\mu\beta} \;=\; Aq^\alpha q^\beta + Cq^\beta q^\alpha + Dq^\alpha(p+p')^\beta + Fq^\beta(p+p')^\alpha = 0$$

So this says that $A = -C$ and $D = F = 0$. We are left with:

$$\begin{aligned}\Gamma^{\alpha\mu\beta} \;=\;& A(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) + Eg^{\alpha\beta}(p+p')^\mu + I(p+p')^\mu(p+p')^\beta q^\alpha \\ &+ J(p+p')^\alpha(p+p')^\mu q^\beta + K(p+p')^\alpha(p+p')^\beta(p+p')^\mu + Lq^\alpha q^\beta(p+p')^\mu\end{aligned}$$

Additionally, all terms with $(p')^\alpha$ and p^β will vanish because of the transversality relation above. This means that we can replace $(p+p')^\alpha$ with q^α and $(p+p')^\beta$ with $-q^\beta$. Doing this we have

$$\begin{aligned}\Gamma^{\alpha\mu\beta} \;=\;& A(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) + Eg^{\alpha\beta}(p+p')^\mu - I(p+p')^\mu q^\beta q^\alpha \\ &+ Jq^\alpha(p+p')^\mu q^\beta - Kq^\alpha q^\beta(p+p')^\mu + Lq^\alpha q^\beta(p+p')^\mu\end{aligned}$$

So the last four terms are the same form! We will write this in the standard form:

$$\Gamma^{\alpha\mu\beta} \;=\; -G_1(q^2)g^{\alpha\beta}(p+p')^\mu - G_2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\mu q^\beta q^\alpha$$

The minus signs in the first two terms are just because of the "mostly minus" metric, and ensures that $G_1(0) = +1$. Also the introduction of the ρ mass in the third term ensures G_3 to be the same dimension as the other two form factors.

4. [20 pts] Calculate the differential cross-section in terms of the form factors for $e^-\rho^+ \to e^-\rho^+$ scattering. For this, you can average/sum over initial/final polarization states of the ρ, and so you need

$$\sum_i \epsilon_\mu^{(i)}(p)\epsilon_\nu^{*(i)}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_\rho^2}$$

where i runs over the *three* spin states of the ρ.

Solution:

So here the matrix element is given by

$$\begin{aligned}
i\mathcal{M} &= (ie)^2\bar{u}^{(r)}(k')\gamma^\nu u^{(s)}(k)\frac{-ig_{\nu\mu}}{q^2}\epsilon_\beta^{(i)}(p)\epsilon_\alpha^{*(j)}(p')\Big[-G_1(q^2)g^{\alpha\beta}(p+p')^\mu - G_2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) \\
&\quad + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\mu q^\beta q^\alpha\Big] \\
&= \frac{ie^2}{q^2}\bar{u}^{(r)}(k')\gamma_\mu u^{(s)}(k)\epsilon_\beta^{(i)}(p)\epsilon_\alpha^{*(j)}(p')\Big[-G_1(q^2)g^{\alpha\beta}(p+p')^\mu - G_2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) \\
&\quad + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\mu q^\beta q^\alpha\Big] \\
\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{6(q^2)^2}\,\mathrm{Tr}[(\not{k}' + m)\gamma_\mu(\not{k} + m)\gamma_\nu]\left[-g_{\beta\beta'} + \frac{p_\beta p_{\beta'}}{m_\rho^2}\right]\left[-g_{\alpha\alpha'} + \frac{p'_\alpha p'_{\alpha'}}{m_\rho^2}\right] \\
&\quad \times\Big[-G_1(q^2)g^{\alpha\beta}(p+p')^\mu - G_2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) \\
&\quad + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\mu q^\beta q^\alpha\Big] \\
&\quad \times\Big[-G_1(q^2)g^{\alpha'\beta'}(p+p')^\nu - G_2(q^2)(g^{\alpha'\nu}q^{\beta'} - g^{\beta'\nu}q^{\alpha'}) \\
&\quad + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\nu q^{\beta'} q^{\alpha'}\Big]
\end{aligned}$$

The factor of $1/6$ comes from averaging the two electron spins and the three rho spins. Now, since the ρ spin sums are symmetric under interchange of indices, we can make a simplification. The $G_{1,3}$ terms are symmetric under $\alpha \leftrightarrow \beta$ interchange, while G_2 is antisymmetric. So taking the first ρ factor with the two spin sum factors, these symmetries are maintained. Thus we can drop all terms that will be proportional to

$$G_1G_2, \qquad G_3G_2$$

So we will write

$$\begin{aligned}
&\Big[-G_1(q^2)g^{\alpha\beta}(p+p')^\mu - G_2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha) + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\mu q^\beta q^\alpha\Big] \\
\times&\Big[-G_1(q^2)g^{\alpha'\beta'}(p+p')^\nu - G_2(q^2)(g^{\alpha'\nu}q^{\beta'} - g^{\beta'\nu}q^{\alpha'}) + \frac{1}{2m_\rho^2}G_3(q^2)(p+p')^\nu q^{\beta'} q^{\alpha'}\Big] \\
&= G_1^2(q^2)g^{\alpha\beta}g^{\alpha'\beta'}(p+p')^\mu(p+p')^\nu + \frac{1}{4m_\rho^4}G_3^2(q^2)(p+p')^\mu(p+p')^\nu q^\beta q^\alpha q^{\beta'} q^{\alpha'} \\
&-\frac{1}{2m_\rho^2}G_1(q^2)G_3(q^2)\left[(p+p')^\mu q^\beta q^\alpha g^{\alpha'\beta'}(p+p')^\nu + g^{\alpha\beta}(p+p')^\mu(p+p')^\nu q^{\beta'} q^{\alpha'}\right] \\
&\qquad +G_2^2(q^2)(g^{\alpha\mu}q^\beta - g^{\beta\mu}q^\alpha)(g^{\alpha'\nu}q^{\beta'} - g^{\beta'\nu}q^{\alpha'})
\end{aligned}$$

And we have just dropped the terms that will vanish when contracting with the spin sums. Let's dot the $\beta\beta'$ spin sum into this first, and we'll call this $L_\rho^{\alpha\alpha'\mu\nu}$. We will consistently use the relation $p^2 = (p')^2 = m_\rho^2$

$$\begin{aligned}
L_\rho^{\alpha\alpha'\mu\nu} &= (p+p')^\mu(p+p')^\nu\left[-G_1^2(q^2)g^{\alpha\alpha'} - \frac{1}{4m_\rho^4}q^2G_3^2(q^2)q^\alpha q^{\alpha'} + \frac{1}{m_\rho^2}G_1(q^2)G_3(q^2)q^{\alpha'}q^\alpha\right] \\
&\quad - G_2^2(q^2)(q^2g^{\alpha\mu}g^{\alpha'\nu} - g^{\alpha\mu}q^{\alpha'}q^\nu - q^\alpha g^{\alpha'\nu}q^\mu + g^{\mu\nu}q^\alpha q^{\alpha'}) \\
&\quad + \frac{(p+p')^\mu(p+p')^\nu}{m_\rho^2}\left[G_1^2(q^2)p^\alpha p^{\alpha'} + \frac{(q\cdot p)^2}{4m_\rho^4}G_3^2(q^2)q^\alpha q^{\alpha'} - \frac{(q\cdot p)}{2m_\rho^2}G_1(q^2)G_3(q^2)\left[q^\alpha p^{\alpha'} + q^{\alpha'}p^\alpha\right]\right] \\
&\quad + \frac{1}{m_\rho^2}G_2^2(q^2)(g^{\alpha\mu}(p\cdot q) - p^\mu q^\alpha)(g^{\alpha'\nu}(p\cdot q) - p^\nu q^{\alpha'})
\end{aligned}$$

Now dotting the other spin sum, dropping the arguments of the form factors:

$$\begin{aligned}
L_\rho^{\mu\nu} &= (p+p')^\mu(p+p')^\nu\Bigg[\left(2 + \frac{(p\cdot p')^2}{m_\rho^4}\right)G_1^2 \\
&\quad + \left(-\frac{q^2}{m_\rho^2} + \frac{(q\cdot p)^2}{m_\rho^4} + \frac{(q\cdot p')^2}{m_\rho^4} - \frac{(q\cdot p)}{m_\rho^6}(q\cdot p')(p\cdot p')\right)G_1G_3 \\
&\quad + \left(\frac{(q^2)^2}{4m_\rho^4} - \frac{(q\cdot p)^2q^2}{4m_\rho^6} - \frac{(q\cdot p')^2q^2}{4m_\rho^6} + \frac{(q\cdot p)^2(q\cdot p')^2}{4m_\rho^8}\right)G_3^2\Bigg] \\
&\quad + G_2^2\Bigg(2(q^2g^{\mu\nu} - q^\mu q^\nu) - \frac{1}{m_\rho^2}\Big((p\cdot q)^2g^{\mu\nu} - (p\cdot q)p^\mu q^\nu - (p\cdot q)q^\mu p^\nu + q^2p^\mu p^\nu \\
&\quad + q^2(p')^\mu(p')^\nu - (q\cdot p')(p')^\mu q^\nu - (q\cdot p')(p')^\nu q^\mu + (q\cdot p')^2g^{\mu\nu}\Big) \\
&\quad + \frac{1}{m_\rho^4}((p')^\mu(p\cdot q) - p^\mu(p'\cdot q))((p')^\nu(p\cdot q) - p^\nu(q\cdot p'))\Bigg)
\end{aligned}$$

We will use the lab frame of the ρ for this calculation, and we will neglect the recoil of the ρ (so we're at some relatively low energy). This means that

$$p \approx p' = (m_\rho, \mathbf{0})$$

so

$$q\cdot p = q\cdot p' \approx 0, \qquad p'\cdot p \approx m_\rho^2$$

This allows us to simplify the above expression significantly. Note, though we don't say $q^2 = 0$, because we will allow the electron to scatter off at some angle, so $q^2 = -2|\mathbf{k}|^2(1-\cos\theta)$:

$$\begin{aligned}
L_\rho^{\mu\nu} &= (p+p')^\mu(p+p')^\nu\left[3G_1^2 - \frac{q^2}{m_\rho^2}G_1G_3 + \frac{(q^2)^2}{4m_\rho^4}G_3^2\right] \\
&\quad + G_2^2\left(2(q^2g^{\mu\nu} - q^\mu q^\nu) - \frac{q^2}{m_\rho^2}\left(p^\mu p^\nu + (p')^\mu(p')^\nu\right)\right)
\end{aligned}$$

Let's dot this into the electron trace, which is

$$\mathrm{Tr}[\not{k}'\gamma_\mu\not{k}\gamma_\nu] = 4[k'_\mu k_\nu + k'_\nu k_\nu - (k'\cdot k)g_{\mu\nu}]$$

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{2e^4}{3(q^2)^2}[k'_\mu k_\nu + k'_\nu k_\nu - (k'\cdot k)g_{\mu\nu}]\Bigg\{(p+p')^\mu(p+p')^\nu\left[3G_1^2 - \frac{q^2}{m_\rho^2}G_1G_3 + \frac{(q^2)^2}{4m_\rho^4}G_3^2\right] \\
&\quad + G_2^2\left(2(q^2g^{\mu\nu} - q^\mu q^\nu) - \frac{q^2}{m_\rho^2}\left(p^\mu p^\nu + (p')^\mu(p')^\nu\right)\right)\Bigg\} \\
&= \frac{2e^4}{3(q^2)^2}\Bigg[2\Bigg\{k'\cdot(p+p')k\cdot(p+p')\left[3G_1^2 - \frac{q^2}{m_\rho^2}G_1G_3 + \frac{(q^2)^2}{4m_\rho^4}G_3^2\right] \\
&\quad + G_2^2\left(2q^2(k'\cdot k) - 2(k\cdot q)(k'\cdot q) - \frac{q^2}{m_\rho^2}\left((p\cdot k')(p\cdot k) + (p'\cdot k')(p'\cdot k)\right)\right)\Bigg\} \\
&\quad - (k'\cdot k)\left\{12m_\rho^2G_1^2 - 4q^2G_1G_3 + \frac{(q^2)^2}{m_\rho^2}G_3^2 + 4q^2G_2^2\right\}\Bigg]
\end{aligned}
$$

For more simplification:

$$k\cdot q = -k'\cdot k = -k'\cdot q$$

Similarly, $k'\cdot p = k'\cdot p'$ and the same for k. We also have

$$k'\cdot p = k\cdot p = k\cdot p' = k'\cdot p' = m_\rho E$$

$$k\cdot k' = -\frac{1}{2}q^2 = 2|\mathbf{k}|^2\sin^2\frac{\theta}{2} = 2E^2\sin^2\frac{\theta}{2}$$

So we have

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{3E^2\sin^4\frac{\theta}{2}}\left(1-\sin^2\frac{\theta}{2}\right)\left[3m_\rho^2G_1^2 - 4E^2\sin^2\frac{\theta}{2}G_1G_3 + 4\frac{E^4}{m_\rho^2}\sin^4\frac{\theta}{2}G_3^2\right]$$

Notice that in this limit of neglecting recoil, there is no contribution from the G_2 form factor (this is connected with the magnetic moment of the ρ, and thus if the ρ is not recoiling, there would be no effect). Now the differential cross-section in the ρ lab frame is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{12m_\rho^2E^2\sin^4\frac{\theta}{2}}\left(1-\sin^2\frac{\theta}{2}\right)\left[3m_\rho^2G_1^2 - 4E^2\sin^2\frac{\theta}{2}G_1G_3 + 4\frac{E^4}{m_\rho^2}\sin^4\frac{\theta}{2}G_3^2\right]$$

5. [30 pts] Now calculate the decay rate for the process $B \to De\bar{\nu}_e$, where B, D are heavy-light spin-zero mesons with quark content $D \sim c\bar{q}, B \sim b\bar{q}$, and $q = u, d, s$. First, we can express the matrix element in terms of two form factors:

$$\langle D(p')|V^\mu|B(P)\rangle = f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu$$

Neglecting the electron mass, we can ignore f_-. The matrix element for this process is given by

$$\mathcal{M}(B\to De\bar{\nu}_e) = \sqrt{2}G_FV_{cb}f_+(q^2)(p+p')^\mu\bar{u}(p_e)\gamma_\mu\frac{1-\gamma_5}{2}v(p_{\nu_e})$$

with G_F the Fermi coupling constant (relevant to the weak interactions), and V_{cb} is a CKM matrix element which describes the likelihood of the $b\to c$ transition. Calculate the decay rate using this, summing over the final spins. To keep the q^2-dependence explicit, calculate $d\Gamma/dq^2$, where

$$\frac{d\Gamma}{dq^2} = d\Gamma\delta[q^2 - (p'-p)^2]$$

The tricky part of this is to calculate the 3-body phase space integrals.

Solution:

We only are summing over the final spins (no averages over the initial spin, because the B is a spin-0 particle)

$$\begin{aligned}
\langle|\mathcal{M}(B\to De\bar{\nu}_e)|^2\rangle &= \frac{1}{2}G_F^2|V_{cb}|^2|f_+(q^2)|^2(p+p')^\mu(p+p')^\nu\,\mathrm{Tr}[(\not{p}_e+m)\gamma_\mu(1-\gamma_5)\not{p}_{\nu_e}(1+\gamma_5)\gamma_\nu] \\
&= G_F^2|V_{cb}|^2|f_+(q^2)|^2(p+p')^\mu(p+p')^\nu\,\mathrm{Tr}[(\not{p}_e+m)\gamma_\mu(1-\gamma_5)\not{p}_{\nu_e}\gamma_\nu] \\
&= G_F^2|V_{cb}|^2|f_+(q^2)|^2(p+p')^\mu(p+p')^\nu\,\mathrm{Tr}[\not{p}_e\gamma_\mu\not{p}_{\nu_e}\gamma_\nu]
\end{aligned}$$

The γ_5 term vanishes because of the $\mu\leftrightarrow\nu$ symmetry.

Now, let's start with this and calculate the differential decay rate is

$$\frac{d\Gamma}{dq^2}=\frac{1}{2m_B}\frac{d^3p'}{(2\pi)^3 2E'}\frac{d^3p_e}{(2\pi)^3 2E_e}\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}(2\pi)^4\delta^4(p-p'-p_e-p_\nu)\delta[q^2-(p'-p)^2]\langle|\mathcal{M}|^2\rangle$$

We first integrate over the electron and neutrino momenta, so we calculate

$$\begin{aligned}
J_{\mu\nu} &= \int\frac{d^3p_e}{(2\pi)^3 2E_e}\int\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}\,\mathrm{Tr}[\not{p}_e\gamma_\mu\not{p}_{\nu_e}\gamma_\nu](2\pi)^4\delta^4(q-p_e-p_\nu) \\
&= \int\frac{d^3p_e}{(2\pi)^3 2E_e}\int\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}\,\mathrm{Tr}[\not{p}_e\gamma_\mu\not{p}_{\nu_e}\gamma_\nu](2\pi)^4\delta^4(q-p_e-p_\nu)
\end{aligned}$$

We can evaluate this easily knowing that it can only be a function of q, so

$$J_{\mu\nu}=Aq_\mu q_\nu+Bg_{\mu\nu}$$

And we know that $q=p'-p=p_e+p_\nu$ dotted into this must make it vanish, since dotting the electron and neutrino momenta into the original expression give us $\not{p}_e^2=0$ (and the same for the neutrino), so $q^2A=-B$, and

$$J_{\mu\nu}=A(q_\mu q_\nu-q^2g_{\mu\nu})$$

Now let's trace $J=J^\mu_\mu$, so we have

$$J=-3q^2A$$

and now the integral itself, we use $\gamma^\mu\not{p}\gamma_\mu=-2\not{p}$ and $q^2=2p_e\cdot p_{\nu_e}$:

$$J=-\int\frac{d^3p_e}{(2\pi)^3 2E_e}\int\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}\,\mathrm{Tr}[\not{p}_e\not{p}_{\nu_e}](2\pi)^4\delta^4(q-p_e-p_\nu)$$

$$J=-4\int\frac{d^3p_e}{(2\pi)^3 2E_e}\int\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}p_e\cdot p_{\nu_e}(2\pi)^4\delta^4(q-p_e-p_\nu)$$

$$J=-\frac{1}{8\pi^2E_e}\int E_\nu dE_\nu d\Omega q^2\delta(q^0-E_e-E_\nu)$$

or

$$J=-\frac{1}{2\pi}q^2$$

so

$$J_{\mu\nu}=\frac{1}{6\pi}(q_\mu q_\nu-q^2g_{\mu\nu})$$

Now we can evaluate (remember $2p\cdot p'=-q^2+m_B^2+m_D^2$:

$$\begin{aligned}
(p+p')^\mu(p+p')^\nu(q_\mu q_\nu-q^2g_{\mu\nu}) &= (q\cdot(p+p')q\cdot(p+p')-q^2(p+p')^2) \\
&= [p^2-(p')^2]^2-q^2(p+p')^2 \\
&= [m_B^2-m_D^2]^2-q^2(m_B^2+m_D^2+2p\cdot p') \\
&= [m_B^2-m_D^2]^2-q^2(2m_B^2+2m_D^2-q^2) \\
&= m_B^4+m_D^4-2m_B^2m_D^2-2q^2m_B^2-2q^2m_D^2+q^4 \\
&= (q^2-m_B^2-m_D^2)^2-4m_B^2m_D^2
\end{aligned}$$

and we have the two-body phase space formula:

$$
\begin{aligned}
\int \frac{d^3p'}{(2\pi)^3 2E'}\delta[q^2-(p'-p)^2] &= \frac{4\pi}{2(2\pi)^3}\int \frac{(p')^2dp'}{E'}\delta[q^2-(p'-p)^2] \\
&= \frac{1}{4\pi^2}\int \frac{(p')^2dp'}{E'}\delta[-m_D^2-m_B^2+2m_BE'+q^2] \\
&= \frac{1}{4\pi^2}\int \frac{[(E')^2-m_D^2]dE'}{\sqrt{(E')^2-m_D^2}}\delta[-m_D^2-m_B^2+2m_BE'+q^2] \\
&= \frac{1}{4\pi^2}\int \frac{[(E')^2-m_D^2]^{1/2}dE'}{2m_B}\delta[E'-(m_D^2+m_B^2-q^2)/2m_B] \\
&= \frac{1}{16\pi^2m_B^2}[((m_D^2+m_B^2-q^2))^2-4m_B^2m_D^2]^{1/2} \\
&= \frac{1}{16\pi^2m_B^2}\sqrt{(q^2-m_B^2-m_D^2)^2-4m_B^2m_D^2}
\end{aligned}
$$

So now we have

$$
\frac{d\Gamma}{dq^2} = \frac{G_F^2|V_{cb}|^2|f_+(q^2)|^2}{192\pi^3m_B^3}[(q^2-m_B^2-m_D^2)^2-4m_B^2m_D^2]^{3/2}
$$

Now, to get the total rate of course we could now integrate over q^2, but this form is useful in Heavy Quark Effective Theory.