Problem Set 7

Due: By classtime, Nov. 12, 2008

1. [20 pts] Using the square-root representation of the non-linear σ model, as defined in Ch. IV of Donoghue *et al.* as (keeping only the most relevant terms)

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu S)^2 - 2\mu^2 S^2\right] + \frac{1}{2}\left(\frac{v+S}{v}\right)^2 \left[(\partial_\mu \varphi_i)^2 + \frac{(\varphi_i \partial_\mu \varphi_i)^2}{v^2 - \varphi_i^2}\right]$$

Here S plays the role of either ρ or σ in our other parametrizations, and in the end we would take $\mu \to \infty$ to decouple it from the theory. Using again the definitions

$$\pi^\pm = \frac{1}{\sqrt{2}}(\varphi_1 \mp i\varphi_2)\ , \qquad \pi^0 = \varphi_3$$

calculate the tree level amplitude for $\pi^+\pi^0 \to \pi^+\pi^0$ scattering. The two diagrams are the same as in class, and why doesn't the the second diagram contribute? You should write down the relevant Feynman rules for the $S\varphi_i\varphi_i$ and the quartic φ_i interactions (be careful, since these will involve momenta).

Solution:

We need to expand the term which is $1/(v^2 - \varphi_i^2)$, but can keep just the leading term in the Taylor expansion, since we only need the quartic term. Expanded the $(v+S)^2$ term and dropping all term that don't contribute (and forgetting the kinetic term for now):

$$\mathcal{L} = \frac{1}{2v^2}(\varphi_i \partial_\mu \varphi_i)^2 + \frac{1}{v} S(\partial_\mu \varphi_i)^2$$

or in terms of the charged fields (keeping just the $\pi^+\pi^-\pi^0\pi^0$ terms):

$$\mathcal{L} = \frac{1}{v^2}\pi^+\partial_\mu\pi^-\pi^0\partial^\mu\pi^0 + \frac{1}{v^2}\pi^-\partial_\mu\pi^+\pi^0\partial^\mu\pi^0 + \frac{1}{v}S(2\partial_\mu\pi^+\partial^\mu\pi^- + \partial_\mu\pi^0\partial^\mu\pi^0)$$

So the vertices for the $S\pi\pi$ vertices are (assuming all momenta are directed inward)

$$-\frac{2i}{v}p^+_\mu p^\mu_-$$

or for the neutral pions

$$-\frac{2i}{v}p^0_\mu p^\mu_{0'}$$

where the momenta have the same assignments as in class. The 4-point coupling is also simple

$$-\frac{i}{v^2}(p_+ + p_-)_\mu (p_0 + p_{0'})^\mu$$

The two diagrams that we have are the same that we had in the linearized form of the σ-model. The diagram with an intermediate S has the form, using the Feynman rules (setting $p_- = -p'_+$ and flipping the sign of p'_0)

$$\frac{2i}{v}p_+\cdot p'_+ \frac{i}{(p'_+ - p_+)^2 - (2\mu)^2}\frac{2i}{v}p_0\cdot p'_0$$

But since we can relate $p_+\cdot p'_+$ and $p_0\cdot p'_0$ to $q^2 = (p'_+ - p_+)^2$, we can see that this is of order q^4, higher order than the other diagram. Additionally we will take $\mu\to\infty$ to decouple the S, and there are no corresponding factors of μ in the numerator to compensate (as there was in the linear σ model), so this term vanishes at this order.

The other diagram gives us

$$-i\mathcal{M} = -\frac{i}{v^2}(p_+ - p'_+)\cdot(p_0 - p'_0)$$

But $q = p'_+ - p_+ = p_0 - p'_0$, so this is just

$$-i\mathcal{M} = +\frac{iq^2}{v^2}$$

As it should be.

2. [10 pts] The nice thing about starting from the QCD Lagrangian is that it easily generalizes to n_f flavors of light quarks. In other words, if we didn't assume that just the up and down quarks were light, we could say we had an approximate

$$SU(n_f)_L \times SU(n_f)_R$$

chiral symmetry. Of course, it would be absurd to think we could treat all six quarks as nearly massless, but it is not much of a stretch to set $n_f = 3$ and include the strange quark. Thus, the required symmetry breaking pattern would be

$$SU(3) \times SU(3)_R \to SU(3)_V$$

This would give rise to 8 Goldstone Bosons, which we can identify as the 8 lightest pseudoscalar mesons: the pions, the kaons, and the η. The corresponding chiral Lagrangian looks exactly the same as the $SU(2)$ case:

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + \frac{\mu f^2}{2}\,\mathrm{Tr}[M\Sigma + \Sigma^\dagger M]$$

but now everything here is a 3×3 matrix. We have $\Sigma = \exp[i\phi^a\lambda^a/f]$, where λ^a are the Gell-Mann matrices, and we have the quark mass matrix $M = \mathrm{diag}(m_u, m_d, m_s)$.

(a) Show that with the definitions

$$\pi^\pm = \frac{1}{\sqrt{2}}(\phi^1 \mp i\phi^2),\ K^\pm = \frac{1}{\sqrt{2}}(\phi^4 \mp i\phi^5),\ K^0 = \frac{1}{\sqrt{2}}(\phi^6 - i\phi^7),\ \overline{K}^0 = \frac{1}{\sqrt{2}}(\phi^6 + i\phi^7),$$

$$\pi^0 = \phi^3, \quad \eta = \phi^8$$

and using the Gell-Mann matrices that

$$\phi^a T^a = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

Solution:

This is just a straightforward exercise.

(b) Notice I have kept the quark masses general. Calculate the masses of the mesons. Note the mass term is not diagonal; there is a $\pi^0\eta$ term. Calculate its coefficient, and what is it in the isospin limit?

Solution:

A straightforward amount of matrix algebra gives us

$$m^2_{\pi^\pm} = m^2_{\pi^0} = 2\mu(m_u + m_d) m^2_{K^\pm} = 2\mu(m_u + m_s) m^2_{K^0,\bar{K}^0} = 2\mu(m_s + m_d)$$

as well as the η mass relationship. Additionally there is a term that mixes the η and π^0, and this term in the Lagrangian is

$$-\frac{\sqrt{3}\mu}{3}(m_d - m_u)\pi^0\eta$$

which vanishes if $m_u = m_d$.

(c) In the isospin limit, give a relation between the π, K, η masses-squared. This is the Gell-Mann–Okubo formula as applied to the octet of mesons (notice that here it involves relations among the squared-masses unlike what we found with the octet of baryons).

Solution:

This is yet another straightforward exercise:

$$m_\eta^2 = \frac{1}{3}(4m_K^2 - m_\pi^2)$$

where we have gotten rid of the charge label on the mesons because they are irrelevant in the isospin limit.

3. [20 pts] Let's now include the weak interactions in χPT. At low energies, we can "integrate out" the W bosons, giving us the Fermi interactions involving four quarks that we briefly discussed in class. One particular interaction that is often relevant is

$$\Delta\mathcal{L}_{(8,1)} = (\overline{s}_L d_L) \sum_{q=u,d,s} (\overline{q}_R q_R)$$

And this is labeled as such because it transforms as an octet (8) under the left-handed part of the $SU(3)$ chiral symmetry, and a singlet (1) under the right-handed part. There of course is an overall coupling constant, but that is irrelevant for this process.

(a) Show that we can write this as

$$\Delta\mathcal{L}_{(8,1)} = (\overline{Q}_L A Q_L)(\overline{Q}_R Q_R)$$

where $Q^T = (u, d, s)$, and determine what the matrix A is.

Solution:

Using

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

we can see that

$$\overline{s}_L d_L = \overline{Q}_L A Q_L$$

(b) We would like to promote A to a spurion field as we did in class. Using the fact that under the $SU(3)_L \times SU(3)_R$ chiral symmetry

$$Q_L \to L Q_L \ , \ Q_R \to R Q_R \ , \ \overline{Q}_L \to \overline{Q}_L L^\dagger \ , \ \overline{Q}_R \to \overline{Q}_R R^\dagger \ ,$$

what would A have to transform like for this term to be invariant under the chiral symmetry?

Solution:

Since

$$Q_L \to L Q_L \ , \ \text{and } \overline{Q}_L \to \overline{Q}_L L^\dagger \ ,$$

Then

$$\overline{Q}_L A Q_L \to \overline{Q}_L L^\dagger A L Q_L$$

so letting

$$A \to L A L^\dagger$$

this is chirally invariant.

(c) There are two independent terms [with different Low-Energy Constants (LEC's)] that arise at the chiral level for this Lagrangian at leading order [that are $O(p^2)$]. Using $\Sigma, \Sigma^\dagger$, the mass matrix $M, M^\dagger$ (which is also a spurion field with the transformation we discussed in class), A, and derivatives what would the chiral level terms be? Remember that the terms must also not be invariant under the parity transformation $\Sigma \to \Sigma^\dagger$ in the same way as the quark-level operator is.

Solution:

Note that if we add a new spurion B to our operator that transforms as

$$B \to RBR^\dagger$$

under the chiral symmetry and

$$B \leftrightarrow A$$

under a parity transformation, the quark level operator given by

$$\Delta\mathcal{L}_{(8,1)} = (\overline{Q}_L A Q_L)(\overline{Q}_R B Q_R)$$

is invariant under parity and the chiral symmetry. So our chiral operators will have one A, one B, and either two derivatives or one factor of the quark mass matrix. In the end we will set $B = I$. Terms that are $O(p^2)$ must have either one factor of M or two derivatives. We can see that the term

$$\partial^\mu \Sigma^\dagger B \partial_\mu \Sigma \to L \partial^\mu \Sigma B \partial_\mu \Sigma^\dagger L^\dagger$$

under a chiral symmetry. Multiplying this on either side by A and then tracing gives us an invariant.

As for terms with the mass matrix, we have

$$M \to LMR^\dagger$$

so

$$MB\Sigma^\dagger \to LMB\Sigma^\dagger L^\dagger$$

and

$$\Sigma B M^\dagger \to L\Sigma B M^\dagger L^\dagger$$

So multiplying these terms with A and tracing gives us two more chiral invariants. These would be

$$\mathrm{Tr}[A\Sigma B M^\dagger], \text{ and } \mathrm{Tr}[AMB\Sigma^\dagger]$$

But with the parity transformation, not only do we interchange A and B and Σ and $\Sigma^\dagger$, but also $M, M^\dagger$, so these two terms must come in with the same overall coefficient.

Combining this with the derivative terms, we find

$$\Delta\mathcal{L}^\chi_{(8,1)} = \alpha_1 \,\mathrm{Tr}[A\partial^\mu \Sigma^\dagger \partial_\mu \Sigma] + \alpha_2 \mu \,\mathrm{Tr}[A(M\Sigma^\dagger + \Sigma M^\dagger)]$$

where $\alpha_{1,2}$ are two new unknown *low energy constants*, just as f, μ are in the original chiral Lagrangian. The μ is just there for convenience.

(d) This Lagrangian is one of the pieces that contributes to the nonleptonic decay of the kaon $K^0 \to \pi^+\pi^-$, for example. Expand the Σ fields to cubic order to determine the Lagrangian in terms of the pions and kaons, and pick out the relevant terms for the $K^0 \to \pi^+\pi^-$ transition, calculate the amplitude and decay rate in terms of the LEC's.

Solution:

We only need to expand Σ and keep cubic terms. First:

$$\Sigma = 1 + \frac{i}{f}\Phi - \frac{1}{2f^2}\Phi^2 - \frac{i}{6f^3}\Phi^3$$

and

$$\partial_\mu \Sigma = \frac{i}{f}\partial_\mu \Phi - \frac{1}{2f^2}\partial_\mu(\Phi^2) - \frac{i}{6f^3}\partial_\mu(\Phi^3)$$

So for the derivative term we can drop the cubic terms, since that would give higher order in Φ terms, and the quadratic terms won't contribute to what we care about, so

$$\alpha_1 \operatorname{Tr}[A\partial^\mu\Sigma^\dagger\partial_\mu\Sigma] = \frac{-i\alpha_1}{2f^3}\left[\operatorname{Tr}[A\partial_\mu\Phi\partial_\mu(\Phi^2)] + \operatorname{Tr}[A\partial_\mu(\Phi^2)\partial_\mu\Phi] + \cdots\right]$$

where the dots refer to any non-cubic in Φ terms. Expanding the derivative terms:

$$\alpha_1 \operatorname{Tr}[A\partial^\mu\Sigma^\dagger\partial_\mu\Sigma] = \frac{-i\alpha_1}{2f^3}\left[\operatorname{Tr}[A\partial_\mu\Phi\partial_\mu\Phi\Phi] + 2\operatorname{Tr}[A\partial_\mu\Phi\Phi\partial_\mu\Phi] + \operatorname{Tr}[A\Phi\partial_\mu\Phi\partial_\mu\Phi]\right]$$

We can use the following form for Φ, since we only care about the $K^0 \to \pi^+\pi^-$ terms:

$$\Phi = \sqrt{2}\begin{pmatrix} 0 & \pi^+ & 0 \\ \pi^- & 0 & K^0 \\ 0 & 0 & 0 \end{pmatrix}$$

(Any terms that would contribute to $\bar{K}^0 \to \pi^+\pi^-$ would be identical to those which contribute to $K^0 \to \pi^+\pi^-$ due to the symmetry of the Lagrangian.)
The trace give us

$$\alpha_1 \operatorname{Tr}[A\partial^\mu\Sigma^\dagger\partial_\mu\Sigma] = \frac{-i\sqrt{2}\alpha_1}{f^3}\left[\pi^-\partial_\mu\pi^+\partial^\mu K^0 + 2\partial_\mu\pi^-\pi^+\partial^\mu K^0 + \partial_\mu\pi^-\partial^\mu\pi^+K^0\right]$$

The mass term is easier, since we just have to keep the cubic terms:

$$\alpha_2\mu \operatorname{Tr}[A(M\Sigma^\dagger + \Sigma M^\dagger)] = \frac{i\alpha_2\mu}{6f^3}\operatorname{Tr}[A(M\Phi^3 - \Phi^3 M)]$$

where

$$\Phi^3 = 2\sqrt{2}\begin{pmatrix} 0 & \pi^+\pi^-\pi^+ & 0 \\ \pi^+\pi^-\pi^- & 0 & \pi^-\pi^+K^0 \\ 0 & 0 & 0 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \pi^-\pi^+K^0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the second equality is because we can ultimately ignore any non-π^2K terms.
Using the mass matrix

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

we have (dropping the terms that are cubic in pion fields)

$$M\Phi^3 - \Phi^3 M = 2\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (m_d - m_s)\pi^-\pi^+K^0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\operatorname{Tr}[A(M\Phi^3 - \Phi^3 M)] = 2\sqrt{2}(m_d - m_s)\pi^-\pi^+K^0$$

Combining this with the derivative term we have finally:

$$\Delta\mathcal{L}^\chi_{(8,1)} = \frac{-i\sqrt{2}\alpha_1}{f^3}\left[\pi^-\partial_\mu\pi^+\partial^\mu K^0 + 2\partial_\mu\pi^-\pi^+\partial^\mu K^0 + \partial_\mu\pi^-\partial^\mu\pi^+K^0\right] + \frac{i\sqrt{2}\alpha_2}{4f^3}\mu(m_d - m_s)\pi^-\pi^+K^0$$

For our amplitude, we need to multiply this by i and convert the derivatives into momenta. We'll define the kaon momentum as going inward, so we'll have derivatives acting on K to become $-ip_K$

and outgoing momenta, the pions, will have $+ip_\pm$ where the sign corresponds to the pion. Thus we get for the amplitude

$$\mathcal{M} = \frac{-i\sqrt{2}\alpha_1}{f^3}\left[p_+ \cdot p_K + 2p_- \cdot p_K - p_- \cdot p_+\right] + \frac{i\sqrt{2}\alpha_2}{4f^3}\mu(m_d - m_s)$$

With momentum conservation:

$$p_K = p_+ + p_-$$

and $p_K^2 = m_K^2$. Plus,

$$m_K^2 = p_K^2 = (p_+ + p_-)^2 = 2m_\pi^2 + 2p_+ \cdot p_-$$

Also, we can use the tree-level mass relations:

$$\mu(m_d - m_s) = \mu(m_d + m_u - m_s - m_u) = m_\pi^2 - m_K^2$$

so we get

$$\mathcal{M} = \frac{-i\sqrt{2}\alpha_1}{f^3}\left[m_\pi^2 + (m_K^2/2 - m_\pi^2) + m_K^2 - 2m_\pi^2 + 2m_\pi^2 - (m_K^2/2 - m_\pi^2)\right] + \frac{i\sqrt{2}\alpha_2}{4f^3}(m_\pi^2 - m_K^2)$$

or

$$\mathcal{M} = \frac{-i\sqrt{2}}{4f^3}\left[4\alpha_1(m_K^2 + m_\pi^2) - \alpha_2(m_\pi^2 - m_K^2)\right]$$

Also, the magnitude of the momentum of either of the pions is given by

$$|\mathbf{p}|^2 = E_\pi^2 - m_\pi^2$$

but $E_\pi = m_K/2$, so

$$|\mathbf{p}| = \sqrt{\frac{m_K^2}{4} - m_\pi^2}$$

And we have

$$\Gamma = \frac{1}{8\pi m_K^2}\sqrt{\frac{m_K^2}{4} - m_\pi^2}\,\frac{2}{16f^6}\left[4\alpha_1(m_K^2 + m_\pi^2) - \alpha_2(m_\pi^2 - m_K^2)\right]^2$$

or

$$\Gamma = \frac{1}{128\pi m_K f^6}\sqrt{1 - \frac{4m_\pi^2}{m_K^2}}\left[4\alpha_1(m_K^2 + m_\pi^2) + \alpha_2(m_K^2 - m_\pi^2)\right]^2$$