Problem Set 8

Due: By classtime, Dec. 19, 2008

We haven't discussed adding nucleons to chiral perturbation theory in class, but it is, for the most part, a straightforward process.

1. We define the nucleon field as before

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

and under the $SU(2)$ group of isospin transformations (which is the vector subgroup of the $SU(2)_L \times SU(2)_R$ chiral symmetry), $N \to VN$, with $V \in SU(2)$. Just as in the case of the heavy-lights, we can define our transformation law under the chiral symmetry as

$$N \to \mathbb{U}(x)N$$

where $\mathbb{U}$ is the same $SU(2)$ matrix which arises in the transformation law of the ξ field we defined in class. Using ξ (or more conveniently, $\mathbb{V}_\mu, \mathbb{A}_\mu$), construct the terms nucleon chiral Lagrangian. Note, without chiral symmetry, we would just have the Lagrangian

$$\mathcal{L}_{\text{nucleons}} = \overline{N}(i\partial\!\!\!/ - M_0)N$$

where M_0 is the leading-order mass of the nucleon, and the kinetic term (as in the case of the heavy-lights) is *not* invariant under the chiral transformation law.

Solution:

Under a chiral transformation we have

$$\overline{N}i\partial\!\!\!/ N \to \overline{N}i\partial\!\!\!/ N + \overline{N}\mathbb{U}^\dagger i(\partial\!\!\!/ \mathbb{U})N$$

and recall that under a chiral transformation (with $\mathbb{V}$ as defined in class):

$$\mathbb{V}_\mu \to \mathbb{U}\mathbb{V}_\mu\mathbb{U}^\dagger + i\mathbb{U}\partial_\mu\mathbb{U}^\dagger$$

so defining a new covariant derivative

$$D_\mu \equiv \partial_\mu + \beta\mathbb{V}_\mu$$

with β a constant to be determined, we can cancel these similar extra terms, so we have

$$\overline{N}iD\!\!\!\!/\,N \to \overline{N}i\partial\!\!\!/ N + \overline{N}\mathbb{U}^\dagger i(\partial\!\!\!/ \mathbb{U})N + i\beta\overline{N}\mathbb{U}^\dagger(\mathbb{U}\gamma^\mu\mathbb{V}_\mu\mathbb{U}^\dagger + i\mathbb{U}\partial\!\!\!/ \mathbb{U}^\dagger)\mathbb{U}N$$

or

$$\overline{N}iD\!\!\!\!/\,N \to \overline{N}iD\!\!\!\!/\,N + \overline{N}\mathbb{U}^\dagger i(\partial\!\!\!/ \mathbb{U})N - \beta\overline{N}\partial\!\!\!/ \mathbb{U}^\dagger\mathbb{U}N$$

Recalling that $\partial\!\!\!/ \mathbb{U}^\dagger\mathbb{U} = -\mathbb{U}^\dagger\partial\!\!\!/ \mathbb{U}$ for any unitary matrix, we see that setting

$$\beta = -i$$

will allow this term to be chirally invariant. Additionally, we need to include the $\mathbb{A}$ term as well. Notice that because $\mathbb{A}(\mathbb{V})$ is an axial-vector (vector), then the former can appear only with the product $\overline{N}\gamma_\mu\gamma_5 N$ to make a Lorentz scalar, while $\mathbb{V}$ must only be able to contract with $\overline{N}\gamma_\mu N$. So our Lagrangian can be written as

$$\mathcal{L}_{\text{nucleons}} = \overline{N}(iD\!\!\!\!/ - M_0)N + ig_A\overline{N}\gamma_\mu\gamma_5\mathbb{A}^\mu N$$

Note that since N is a column vector in isospin space as well as spin space, and $\overline{N}$ is a row vector, this is a scalar in Lorentz, Dirac, and flavor space.

2. Expand the ξ fields to linear order in the pion (note, we're discussing $SU(2)$ here, so use the appropriate version of ξ), and ultimately write the $\mathbb{A}_\mu$ term with $\pi^\pm, \pi^0$. There should be one term with π^+, one with π^-, and two with π^0. Show that you obtain precisely the Lagrangian from Problem Set 5 to this order.

Solution:

The $\mathbb{V}$ term only has even powers of pion fields, and $\mathbb{A}$ has odd powers. We can expand this as

$$\mathbb{A}_\mu = -\frac{1}{2f}\partial_\mu \pi^a \sigma^a + \dots$$

or

$$\mathbb{A}_\mu = -\frac{1}{2f}\begin{pmatrix} \partial_\mu \pi^0 & \sqrt{2}\partial_\mu \pi^+ \\ \sqrt{2}\partial_\mu \pi^- & -\partial_\mu \pi^0 \end{pmatrix}$$

and

$$\overline{N}\gamma_\mu\gamma_5 \mathbb{A}^\mu N = -\frac{1}{2f}(\overline{p}, \overline{n})\gamma_\mu\gamma_5 \begin{pmatrix} \partial_\mu \pi^0 & \sqrt{2}\partial_\mu \pi^+ \\ \sqrt{2}\partial_\mu \pi^- & -\partial_\mu \pi^0 \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix}$$

or

$$\overline{N}\gamma_\mu\gamma_5 \mathbb{A}^\mu N = -\frac{1}{2f}\left[(\overline{p}\gamma_\mu\gamma_5 p - \overline{n}\gamma_\mu\gamma_5 n)\partial_\mu\pi^0 + \sqrt{2}\overline{p}\gamma_\mu\gamma_5 n \partial_\mu \pi^- + \sqrt{2}\overline{n}\gamma_\mu\gamma_5 p\partial_\mu\pi^+\right]$$

So now, we want to relate this to the Lagrangian from before, and to do that we could either calculate all tree-level processes in both theories and show that they are identical, or we can do the following. If we integrate these terms by parts (note, the other terms automatically give us the same kinetic terms), then we get twice as many terms (with an overall minus sign), with derivatives acting either on $n, p, \overline{p}, \overline{n}$. Using the equations of motion, which are going to be

$$\partial\!\!\!/ n = M_0 n + O(\pi)$$

$$\partial_\mu \overline{n}\gamma^\mu = M_0 \overline{n} + O(\pi)$$

and the same for p, we can remove the derivatives from the pions and the γ^μ's by giving a $-M_0$ instead, so the interaction term is (neglecting terms that would be higher order in the pion fields)

$$\overline{N}\gamma_\mu\gamma_5 \mathbb{A}^\mu N = +\frac{M_0}{2f}\left[(\overline{p}\gamma_5 p - \overline{n}\gamma_5 n)\pi^0 + \sqrt{2}\overline{p}\gamma_5 n\pi^- + \sqrt{2}\overline{n}\gamma_5 p\pi^+\right]$$

So if we make the identification

$$g_A^{\text{prob. set 5}} = -g_A^{\text{here}}\frac{M_0}{2f}$$

These two Lagrangians are the same. Notice that it is isospin symmetry which dictates the relative coefficients of the terms.

3. On a different note, in class I showed the rate of decay for $D^{*+} \to D\pi^+$ to be

$$\Gamma = \frac{g_\pi^2 |\mathbf{p}_\pi|^2}{12\pi f^2}$$

What would it be for $D^{*0} \to D^0\pi^0$? Think of the quark assignments

$$D^{*+} \sim c\bar{d}\ , \quad D^{*0} \sim c\bar{u}\ , \quad D^+ \sim c\bar{d}\ , \quad D^0 \sim c\bar{u}\ , \quad \pi^+ \sim u\bar{d}\ , \quad \pi^0 \sim u\bar{u} + d\bar{d}$$

Solution:

This can be done two ways. For one, we can see that in the Nucleon Lagrangian, the vertices with neutral pions are $1/\sqrt{2}$ times those with charged pions. If we expand the heavy-light Lagrangian, we would see the same effect, so the neutral decay would be one-half the charged decay.

Also we can relate the two processes by an isospin decomposition (note this is actually the same method as above, since we used isospin to construct the effective theory). The D mesons are isopsin-1/2 particles, with the D^+ the +1/2 projection and the D^0 the -1/2 projection. This holds for either the pseudoscalar or vector. And we remember that the pions are isospin-1 objects. We can decompose the pure isospin final states as follows, realizing we are adding an $I = 1/2$ and an $I = 1$ state. First, though, since our initial state has total isospin 1/2, the final state must also have $I = 1/2$, since isospin is conserved. Adding $I = 1, 1/2$ to get a state with total isospin of 1/2, we have the decomposition:

$$
\begin{aligned}
|I = 1/2, I_3 = 1/2\rangle &= \frac{1}{\sqrt{3}}(\sqrt{2}D^0\pi^+ - D^+\pi^0) \\
|I = 1/2, I_3 = -1/2\rangle &= \frac{1}{\sqrt{3}}(\sqrt{2}D^+\pi^- + D^0\pi^0)
\end{aligned}
$$

So as long as we assume isospin symmetry, the two decay amplitudes will be related by the factor of $\sqrt{2}$, and thus the decay rate $D^{*0} \to D^0\pi^0$ is

$$\Gamma = \frac{g_\pi^2|\mathbf{p}_\pi|^2}{24\pi f^2}$$